Problem 4. The number $N = \frac{1}{2}ab(a^4 + b^4)$, where a, b are positive integers such that

$$a^4 + b^4 = 1 + ab(1 + 2 + 3 + \ldots + (a + b)).$$

What is N?

Solution. We will show that $N = 2022$. Recall that $1 + 2 + \ldots + n = n(n + 1)/2$. Thus our condition can be stated as

$$a^4 + b^4 = 1 + ab(a + b)(a + b + 1)/2. \tag{1}$$

Setting $u = a + b$, $w = a - b$, we see that $a = (u + w)/2$, $b = (u - w)/2$ so (1) is equivalent to

$$(u + w)^4 + (u - w)^4 = 16 + 2(u^2 - w^2)u(u + 1). \tag{2}$$

Using the binomial formula to expand the left hand side of (2) we get

$$2u^4 + 12u^2w^2 + 2w^4 = 16 + 2u^4 + 2u^3 - 2w^2(u^2 + u). \tag{3}$$

which is equivalent to

$$w^2(w^2 + 7u^2 + u) = 8 + u^3. \tag{4}$$

Note that $w \neq 0$, as otherwise we would have $u^3 = -2$, which in not possible as $u = a + b > 0$. It follows that $w^2 \geq 1$.

Suppose that $w^2 > 1$. Then $w^2 \geq 4$ and $w^2(u + w^2) > 8$. It follows from (4) that $7u^2w^2 < u^3$, i.e.

$$7w^2 < u. \tag{5}$$

Using (4) again, we see that

$$u^3 < u^3 + 8 = w^2(w^2 + 7u^2 + u) < w^2(\frac{u}{7} + 7u^2 + u) = 7w^2u(u + \frac{8}{49})$$

which implies that

$$u - 1 < \frac{u^2}{u + \frac{8}{49}} < 7w^2.$$

By (5), we have $u - 1 < 7w^2 < u$. This however is not possible since u and $7w^2$ are integers. We see that the assumption $w^2 > 1$ leads to a contradiction. Thus $w^2 = 1$ and (4) yields $7u^2 + u + 1 = 8 + u^3$, i.e. $(u^2 - 1)(u - 7) = 0$. Since $u = a + b \geq 2$, we have $u = 7$. Thus $u = 7$ and $w^2 = 1$. If $w = 1$, we get $a = 4, b = 3$ and if $w = -1$ we get $a = 3, b = 4$. In both cases, we have $N = 2022$.

Remark. I created this problem as a new year puzzler to welcome the year 2022.

As a good exercise to practice the above techniques we suggest the following

Problem. Find all integers a, b which satisfy (1).