Problem 6. Prove that the inequality

$$\prod_{i=1}^{n}\prod_{j=1}^{n}(1+|a_i+a_j|) \geq \prod_{i=1}^{n}\prod_{j=1}^{n}(1+|a_i-a_j|)$$

holds for any real numbers $a_1,\ldots,a_n$.

Solution. Taking logarithms of both sides, and setting $f(x)=\ln(1+x)$, our inequality is equivalent to the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(|a_i+a_j|) \geq \sum_{i=1}^{n}\sum_{j=1}^{n} f(|a_i-a_j|)$$

We will prove this inequality by induction on n. Before we start the induction let us record the following 3 properties of the function f:

1. $f(x)$ is increasing.
2. $f:[0,\infty)\longrightarrow \mathbb{R}$ is continuous.
3. If $0<a\leq x$ then $f(x+a)+f(x-a)<2f(x)$.

To justify the third property note that

$$\ln(1+x+a)+\ln(1+x-a)=\ln((1+x)^2-a^2)<\ln((1+x)^2)=2\ln(1+x).$$

Note that property 1 implies the following simple observation:

Lemma 1. The inequality in the problem is true when all the numbers a_i are either non-negative or non-positive.

Indeed, in this case we have $|a_i+a_j|\geq|a_i-a_j|$ for every i,j, so $f(|a_i+a_j|)\geq f(|a_i-a_j|)$.

We are ready to start our induction. When $n=1$ our inequality says

$$f(2|a_1|)\geq f(0)$$

and this is true by property 1 of f. Suppose that the inequality is true for every $n<N$, for some integer $N\geq 2$. We claim that then we have the following partial rsult:

Lemma 2. If $a_1,a_2,\ldots,a_N$ are real numbers such that for some i,j we have $a_i+a_j=0$ ($i=j$ is allowed) then the inequality holds for $a_1,a_2,\ldots,a_N$.

Indeed, if $a_i=0$ for some i (i.e. $a_i+a_i=0$) then we may assume that $i=N$ and then our inequality takes the following form:

$$\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} f(|a_i+a_j|)+2\sum_{i=1}^{N-1} f(|a_i|)+f(0)\geq \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} f(|a_i-a_j|)+2\sum_{i=1}^{N-1} f(|a_i|)+f(0)$$

which is true by the inductive assumption. Similarly, if $a_i+a_j=0$ for some $i\neq j$ then we may assume $i=N-1$, $j=N$. Thus $a_{N-1}=-a_N=a$ and our inequality takes the following form:

$$\sum_{i=1}^{N-2}\sum_{j=1}^{N-2} f(|a_i+a_j|)+\sum_{i=1}^{N-2} f(|a_i+a|)+\sum_{i=1}^{N-2} f(|a_i-a|)+2f(0)+2f(2|a|)\geq$$

$$\sum_{i=1}^{N-2}\sum_{j=1}^{N-2} f(|a_i-a_j|)+\sum_{i=1}^{N-2} f(|a_i-a|)+\sum_{i=1}^{N-2} f(|a_i+a|)+2f(2|a|)+2f(0)$$

which is equivalent to

$$\sum_{i=1}^{N-2}\sum_{j=1}^{N-2} f(|a_i+a_j|)\geq \sum_{i=1}^{N-2}\sum_{j=1}^{N-2} f(|a_i-a_j|)$$

and the last inequality is true by our inductive assumption.

Suppose now that our inequality is false for some numbers $a_1, \dots, a_N$. We will show that this assumption leads to a contradiction. By Lemma 1 above, among the numbers a_i we must have a positive number and a negative number. Consider the following function

$$H(t) = \sum_{i=1}^{N}\sum_{j=1}^{N} f(|(a_i + t) + (a_j + t)|) - \sum_{i=1}^{N}\sum_{j=1}^{N} f(|a_i - a_j|).$$

Thus $H(t) \geq 0$ if and only if our inequality is true for the numbers $a_1+t, \dots, a_N+t$. By our assumption, $H(0) < 0$. If $t = -a_i$ then $H(t) \geq 0$ by Lemma 2. Suppose that $a_i > 0$ and $a_j < 0$. Thus H is non-negative at the ends of the interval $[-a_i, -a_j]$ and it is negative at 0. Since H is continuous (property 2 of f), at some point $s \in (-a_i, -a_j)$ the function H assumes its smallest value on $[-a_i, -a_j]$. Since $H(s) < 0$, none of the numbers $a_i + s + b_j + s = a_i + b_j + 2s$ is zero by Lemma 2. Let $u > 0$ be smaller than all the numbers $|a_i + b_j + 2s|/2$ and such that $s \pm 2u \in (-a_i, -a_j)$ (any sufficiently small positive number has these properties). Then $H(s-u) \geq H(s)$ and $H(s+u) \geq H(s)$. Note that since $2u > 0$ is smaller than all the numbers $|a_i + b_j + 2s|$, among the two numbers $|a_i + b_j + 2(s-u)|$, $|a_i + b_j + 2(s+u)|$ one is equal to $|a_i + b_j + 2s| - 2u$ and the other to $|a_i + b_j + 2s| + 2u$. It follows that

$$f(|(a_i + s + u) + (a_j + s + u)|) + f(|(a_i + s - u) + (a_j + s - u)|) =$$

$$f(|(a_i + s) + (a_j + s)| + 2u) + f(|(a_i + s) + (a_j + s)| - 2u) < 2f(|(a_i + s) + (a_j + s)|),$$

where the last inequality follows from property 3 of f. Adding these inequalities for each pair i, j we see that

$$H(s+u) + H(s-u) < 2H(s).$$

This however contradicts the fact that both $H(s+u)$ and $H(s-u)$ are no smaller than $H(s)$. The contradiction shows that our assumption that the inequality in question is not always true is false, which completes the proof of the inductive step.

Remark. Adjust the proof above to show that if our inequality becomes equality then $a_i + a_j = 0$ for some i, j. Then solve the following problem.

Problem. When does the inequality of the problem become an equality?

Remark. Note that property 3 of our function f means that f is J-concave (or Jensen concave) on $[0, \infty)$ (which is the same as saying that $-f$ is J-convex; see the discussion at the end of the solution to Problem 3).

Problem. Prove that if a function $f : [0, \infty) \longrightarrow \mathbb{R}$ has properties 2 and 3 above and it is bounded below then it also has property 1.

Problem. Let $\alpha \in (0, 1]$. Prove the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n} |a_i + a_j|^{\alpha} \geq \sum_{i=1}^{n}\sum_{j=1}^{n} |a_i - a_j|^{\alpha}.$$

(This is also true for $\alpha \in (1, 2]$, but a different technique is needed to prove it).