Problem 4. A sequence (a_n) of positive integers has been created using the following process:

$$a_{n+1} = a_n + 3\frac{a_n}{p_n}$$

where p_n is a prime divisor of a_n. Prove that there is a positive integer k such that the equality $a_{n+k} = 2ka_n$ holds for infinitely many values of n.

Solution. Any integer m can be written in a unique way as bc, where all prime divisors of b are among $\{2, 3, 5\}$ and all prime divisors of c are bigger than 5. We will call a the small divisor of m and b the large divisor of m.

Write $a_n = b_n c_n$, where b_n is the small divisor of a_n and c_n is the large divisor of a_n. Note that if $p_n > 5$ then $p_n | c_n$ and $p_n + 3$ is even. Since $a_{n+1} = 2b_n \dfrac{c_n}{p_n} \dfrac{p_n + 3}{2}$, we see than the large divisor c_{n+1} of a_{n+1} is at most $\dfrac{c_n}{p_n} \dfrac{p_n + 3}{2}$. It follows that $c_{n+1} < c_n$ if $p_n > 5$.

If $p_n = 2$ then $a_{n+1} = 5\frac{b_n}{2} c_n$, so $b_{n+1} = 5\frac{b_n}{2}$ and $c_{n+1} = c_n$.

If $p_n = 3$ then $a_{n+1} = 6\frac{b_n}{3} c_n = 2b_n c_n$, so $b_{n+1} = 2b_n$ and $c_{n+1} = c_n$.

If $p_n = 5$ then $a_{n+1} = 8\frac{b_n}{5} c_n$, so $b_{n+1} = 8\frac{b_n}{5}$ and $c_{n+1} = c_n$.

It follows that the sequence c_n is a non-decreasing sequence of positive integers and therefore it must be constant from some point on: there is N such that $c_N = c_{N+1} = c_{N+2} = \ldots$. Furthermore, $p_n \in \{2, 3, 5\}$ for all $n \geq N$ (as otherwise we would have $c_{n+1} < c_n$).

Observe now that if $p_n = 3$ then $a_{n+1} = 2a_n$. Thus, if $p_n = 3$ for infinitely many n then $k = 1$ works.

Suppose now that $p_n = 3$ for only infinitely many n. Thus there is M such that $p_n \in \{2, 5\}$ for all $n \geq M$. Note that if $p_n = \ldots = p_{n+k} = p$ then p^{k+1} divides a_n. This implies that $p_n = 2$ must happen for infinitely many n. Otherwise we would have m such that $p_n = 5$ for all $n \geq m$ and a_m would be divisible by every power of 5, which is not possible. Similarly $p_n = 5$ must happen for infinitely many n. It follows that there are infinitely many values of n such that $p_n = 2$ and $p_{n+1} = 5$. For such n we have $a_{n+2} = 4a_n$. Thus $k = 2$ works in this case.