Problem 3. a) Is there a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right) + \sin^2(x-y)$$

for all $x, y \in \mathbb{R}$?

b) Is there a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right) + \sin|x-y|$$

for all $x, y \in \mathbb{R}$?

Solution. a) An example of such a function is $f(x) = 4x^2$. Indeed, we have

$$\frac{f(x)+f(y)}{2} = 2(x^2+y^2) = (x+y)^2 + (x-y)^2 = f\left(\frac{x+y}{2}\right) + (x-y)^2 \geq f\left(\frac{x+y}{2}\right) + \sin^2(x-y),$$

(the last inequality follows from the fact that $|\sin u| \leq |u|$ for every real number u, hence $(x-y)^2 \geq \sin^2(x-y)$).

b) We will show that no such function exists. Suppose contrary, that f exists. Taking $y = x + a$, we see that

$$\frac{f(x+a)+f(x)}{2} \geq f\left(x + \frac{a}{2}\right) + \sin|a|$$

for all $x, a \in \mathbb{R}$. This is equivalent to

$$f(x+a) - f(x) \geq 2\left(f\left(x+\frac{a}{2}\right) - f(x)\right) + 2\sin|a|$$

for all $x, a \in \mathbb{R}$. Taking $a \neq 0$ and dividing by $|a|$ we get

$$\frac{f(x+a)-f(x)}{|a|} \geq \frac{f\left(x+\frac{a}{2}\right)-f(x)}{\frac{|a|}{2}} + 2\frac{\sin|a|}{|a|}. \tag{1}$$

If f was differentiable at x we could get now a contradiction as follows: let $a > 0$ tend to 0 so we have

$$f'(x) = \lim_{a\to 0^+} \frac{f(x+a)-f(x)}{|a|} \geq \lim_{a\to 0^+} \frac{f\left(x+\frac{a}{2}\right)-f(x)}{\frac{|a|}{2}} + 2\lim_{a\to 0^+}\frac{\sin|a|}{|a|} = f'(x) + 2,$$

which is clearly false. Unfortunately, we can not assume differentiability of f at some point, so we need to modify the argument. Consider x as fixed and set $g(a) = (f(x+a) - f(x))/|a|$. Then (1) takes the form

$$g(a) \geq g\left(\frac{a}{2}\right) + 2\frac{\sin|a|}{|a|}$$

for all $a \neq 0$. It follows that

$$g\left(\frac{a}{2}\right) \geq g\left(\frac{a}{4}\right) + 2\frac{\sin\frac{|a|}{2}}{\frac{|a|}{2}}, \quad g\left(\frac{a}{4}\right) \geq g\left(\frac{a}{8}\right) + 2\frac{\sin\frac{|a|}{4}}{\frac{|a|}{4}}, \quad g\left(\frac{a}{8}\right) \geq g\left(\frac{a}{16}\right) + 2\frac{\sin\frac{|a|}{8}}{\frac{|a|}{8}}, \quad \ldots$$

Putting these inequalities together, we conclude that for any $a \neq 0$ and any positive integer k we have

$$g(a) \geq g\left(\frac{a}{2^k}\right) + 2\left(\frac{\sin|a|}{|a|} + \frac{\sin\frac{|a|}{2}}{\frac{|a|}{2}} + \ldots + \frac{\sin\frac{|a|}{2^{k-1}}}{\frac{|a|}{2^{k-1}}}\right). \tag{2}$$

Note now that for any $u \neq 0$ we have

$$g(u) + g(-u) = \frac{f(x+u)+f(x-u)-2f(x)}{|u|} \geq \frac{2f\left(\frac{(x+u)+(x-u)}{2}\right) + 2\sin|2u| - 2f(x)}{|u|} = 2\frac{\sin 2|u|}{|u|}.$$

Therefore from (2) we get that

$$g(a)+g(-a)\ge g\left(\frac{a}{2^k}\right)+g\left(\frac{-a}{2^k}\right)+4\left(\frac{\sin|a|}{|a|}+\frac{\sin\frac{|a|}{2}}{\frac{|a|}{2}}+\ldots+\frac{\sin\frac{|a|}{2^{k-1}}}{\frac{|a|}{2^{k-1}}}\right)\ge$$

$$2\frac{\sin 2\frac{|a|}{2^k}}{\frac{|a|}{2^k}}+4\left(\frac{\sin|a|}{|a|}+\frac{\sin\frac{|a|}{2}}{\frac{|a|}{2}}+\ldots+\frac{\sin\frac{|a|}{2^{k-1}}}{\frac{|a|}{2^{k-1}}}\right).$$

To get a contradiction it suffices to observe that the right hand side of the last inequality tends to $+\infty$ when k tends ∞, which is a simple consequence of the fact that

$$\lim_{k\to\infty}\frac{\sin\frac{|a|}{2^k}}{\frac{|a|}{2^k}}=1.$$

Indeed, this means that $g(a)+g(-a)$ is $+\infty$, which is an absurd. The contradiction shows that f does not exists.

Remark. Functions $f:(a,b)\longrightarrow\mathbb{R}$ which satisfy

$$\frac{f(x)+f(y)}{2}\ge f\left(\frac{x+y}{2}\right)$$

for all $x,y\in(a,b)$ are called **J-convex** (or Jensen convex). Such functions have been extensively studied. The following exercise collects some key properties of J-convex functions.

Exercise. Let $f:(a,b)\longrightarrow\mathbb{R}$ be a function.

1. Prove that if f is J-convex, then

$$\frac{f(x_1)+f(x_2)+\ldots+f(x_{2^k})}{2^k}\ge f\left(\frac{x_1+x_2+\ldots+x_{2^k}}{2^k}\right)$$

for any positive integer k and any $x_1,\ldots,x_{2^k}\in(a,b)$.

2. Use 1. to show that if f is J-convex, then

$$\frac{f(x_1)+f(x_2)+\ldots+f(x_n)}{n}\ge f\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)$$

for any positive integer n and any $x_1,\ldots,x_n\in(a,b)$. This inequality is called **Jensen's inequality**.

3. Show that if f is J-convex and p,q are non-negative rational numbers such that $p+q=1$ then

$$pf(x)+qf(y)\ge f(px+qy)$$

for any $x,y\in(a,b)$.

4. Show that f is continuous and J-convex if and only if

$$pf(x)+qf(y)\ge f(px+qy)$$

for any $x,y\in(a,b)$ and any non-negative real numbers p,q such that $p+q=1$.

Continuous J-convex functions are called **convex** or **concave up**.

5. Show that if f is differentiable on (a,b) then f is J-convex if and only if the derivative $f'(x)$ is non-decreasing on (a,b).

6. From 5, the functions $f(x)=-\ln x$ and $f(x)=x^\alpha$ for $\alpha\ge 1$ are convex. What does Jensen's inequality tell us when applied to these functions?

Remark. One can ask whether discontinuous J-convex functions exist. The answer is positive, but any such function must be very "pathological" and no effective example of such functions exists. This question is related to a more fundamental question: do discontinuous additive functions exist? Recall that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is called **additive** if $f(x + y) = f(x) + f(y)$ for all x, y. It is not hard to show that a continuous additive function is of the form $f(x) = cx$ for some constant c. Clearly any additive function is J-convex. For many years the existence of discontinuous additive functions was an open problem. It was G. Hamel who first succeeded to show that such functions exist. The argument is quite simple and accessible to anyone with basic knowledge of linear algebra.

The field $\mathbb{R}$ of real numbers can be considered as a vector space over the field $\mathbb{Q}$ of rational numbers. A fundamental result in the theory of vector spaces says that every vector space has a basis (note that for vector spaces which are not finite dimensional, there are some subtle set-theoretical facts needed for the existence of a basis). In particular, there is a basis of $\mathbb{R}$ considered as $\mathbb{Q}$-vector space. Any such basis is called nowadays a **Hamel basis** of $\mathbb{R}$. Note that no constructive example of such basis exist.

Now let H be a Hamel basis of $\mathbb{R}$. Another simple fact from linear algebra is the following: if V is a vector space with a basis B and W is another vector space, then any function $F : B \longrightarrow W$ extends in a unique way to a linear function $F : V \longrightarrow W$. In particular, any function $f : H \longrightarrow \mathbb{R}$ extends to a $\mathbb{Q}$-linear function $f : \mathbb{R} \longrightarrow \mathbb{R}$. For example, we could pick one element $h \in H$ and send it to 1 and all the other elements from H could be sent to 0. The resulting $\mathbb{Q}$-linear function is a discontinuous additive function.

The following (challenging) problem explains the meaning of "pathological" for discontinuous additive functions.

Problem. Prove that if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is additive and not continuous then the graph of f, i.e. the set $\{(x, f(x)) : x \in \mathbb{R}\}$ is dense in the plane $\mathbb{R} \times \mathbb{R}$.

A similar, but slightly more complicated to formulate, "pathological" behavior happens for any discontinuous J-convex function.