Problem 2. Let Γ be the set of all points (a, b) on the cartesian plane such that a, b are positive integers not exceeding 100. A subset H of Γ is called **rounded** if for any two points (a, b) and (A, B) in H, either $a > A - 10$ and $b > B - 10$ or $A > a - 10$ and $B > b - 10$. What is the largest possible size of a rounded subset of Γ.

Solution. Let H be a rounded subset of Γ. Let Γ_m consists of those points (a, b) in Γ such that $a + b = m$. Suppose $(a_1, b_1), \ldots, (a_t, b_t)$ be points in $H \cap \Gamma_m$ for some m, where $a_1 < \ldots < a_t$. It follows that $b_1 > \ldots > b_t$. Since the numbers a_i, b_i are integers, we have $a_1 \leq a_t - t + 1$ and $b_t \leq b_1 - t + 1$. On the other hand, since H is rounded, we have either $a_1 > a_t - 10$ or $b_t > b_1 - 10$. This tells us that $t \leq 10$. In other words, for every m the set $H \cap \Gamma_m$ has at most 10 points, i.e. $|H \cap \Gamma_m| \leq \min(10, |\Gamma_m|)$. Note that

1. Γ_m is non-empty iff $2 \leq m \leq 200$;
2. $|\Gamma_m| = m - 1$ for $1 \leq m \leq 10$;
3. $|\Gamma_m| = 201 - m$ for $192 \leq m \leq 200$;
4. $|\Gamma_m| \geq 10$ for $11 \leq m \leq 191$.

Thus

$$|H| = \sum_{m=2}^{200} |H \cap \Gamma_m| \leq \sum_{m=2}^{200} \min(10, |\Gamma_m|) = 2(1 + 2 + \ldots + 9) + 181 \cdot 10 = 1900. \qquad (1)$$

This proves that any rounded subset of Γ has no more than 1900 elements.

The above discussion also suggests how to construct a rounded set with 1900 elements: we just need a rounded set H such that $|H \cap \Gamma_m| = \min(10, |\Gamma_m|)$ for every m. Consider the subset H of Γ which consists of all points $(a, b) \in \Gamma$ such that either $a \leq 10$ or $b \geq 91$. We claim that H is rounded. Indeed, suppose that (a, b) and (A, B) are in H, $a \leq A$.

- If $A \leq 10$ then $a > A - 10$ and $A > a - 10$. Clearly, either $b > B - 10$ or $B > b - 10$.
- If $A > 10$ then $B \geq 91$, so $A > a - 10$ and $B > b - 10$.

This shows that H is rounded.

Note that Γ has 100^2 elements. The complement of H in Γ consists of points (a, b) such that $10 < a \leq 100$ and $1 \leq b < 91$. There are 90^2 such points. Thus $|H| = 100^2 - 90^2 = 1900$. (Alternatively, one can easily check that $|H \cap \Gamma_m| = \min(10, |\Gamma_m|)$ for every m and use (1)).