Problem 5. We call a positive integer N **prosperous** if

$$\phi(N) + \sigma(N) = 2(N+1) \text{ and } \phi(N)\sigma(N) = (N-5)(N+3).$$

Knowing that both N and $N-504$ are prosperous, find N.
Remark. Here ϕ is the Euler function:

$$\phi(N) = \text{ the number of positive integers which are relatively prime to } N \text{ and do not exceed } N,$$

and $\sigma(N)$ is the sum of all positive divisors of N. These functions are studied in elementary number theory (a topic of Math 407). They both are so called **multiplicative functions**: for any two **relatively prime** integers M, N we have $f(MN) = f(M)f(N)$, where f is either ϕ or σ.

First solution, submitted by Ashton Keith. Using the identity $(a-b)^2 = (a+b)^2 - 4ab$ with $a = \phi(N)$ and $b = \sigma(N)$ we see that if N is prosperous then

$$(\phi(N) - \sigma(N))^2 = (2(N+1))^2 - 4(N-5)(N+3) = 16(N+4).$$

It follows that if N is prosperous then $N+4$ is a square of an integer. Assuming that N and $N-504$ are prosperous, we have $N = a^2 - 4$ and $N - 504 = b^2 - 4$ for some positive integers a, b. Subtracting these equalities we get $504 = a^2 - b^2 = (a-b)(a+b)$. Note that $a-b$ and $a+b$ are of the same parity and 504 is even, so both $a-b$ and $a+b$ are even. There are only six ways of factoring 504 as a product of two even numbers:

$$504 = 2 \cdot 252 = 4 \cdot 126 = 6 \cdot 84 = 12 \cdot 42 = 14 \cdot 36 = 18 \cdot 28.$$

Setting the first factor to be $a-b$ and the second $a+b$ and solving for a, b yields the value of (a, b) to be $(127, 125), (65, 61), (45, 39), (27, 15), (25, 11), (23, 5)$ respectively. Now it is straightforward to verify that out of these six possibilities only the case $a = 45$, $b = 39$ yields both $N = a^2 - 4 = 2021$ and $N - 504 = b^2 - 4 = 1517$ prosperous. Thus $N = 2021$.

Remark. The above ideas can be extended to characterize prosperous numbers as follows. Note that if $a + b = s$ and $ab = t$ then $(x-a)(x-b) = x^2 - sx + t$, i.e. a, b are the roots of the quadratic equation $x^2 - sx + t = 0$. It follows that if N is prosperous then $\phi(N)$ and $\sigma(N)$ are the roots of $x^2 - 2(N+1)x + (N-5)(N+3) = 0$. The discriminant of this equation is $4(N+4)$ so its roots are $N + 1 \pm 2\sqrt{N+4}$. Thus, N is prosperous if and only if $\phi(N) = N + 1 - 2\sqrt{N+4}$ and $\sigma(N) = N + 1 + 2\sqrt{N+4}$. In particular, if N is prosperous then $\sqrt{N+4}$ is a rational number, hence an integer (can you justify this?). In other words, $N = a^2 - 4 = (a-2)(a+2)$ for some integer a (as observed in the first solution). Moreover $\sigma(N) = N + 1 + 2a$. On the other hand, $1, a-2, a+2, N$ are distinct divisors of N (note that $a-2$ can not be 1). Thus $\sigma(N) \geq 1 + (a-2) + (a+2) + N = N + 1 + 2a$ and the equality holds if and only if $1, a-2, a+2, N$ are the only divisors of N. It is easy to see that this implies that both $a-2$ and $a+2$ are prime numbers. In other words, if N is prosperous then $N = p(p+4)$ where both p and $p+4$ are prime numbers. It is easy to see that the converse also holds: if p and $p+4$ are prime then $p(p+4)$ is prosperous.

Second Solution (after Maxwell T Meyers). Recall that if p is a prime and $k > 0$ an integer then $\phi(p^k) = p^k - p^{k-1}$. Since ϕ is multiplicative, if $N = p_1^{k_1} p_2^{k_2} \dots p_s^{k_s}$ then

$$\phi(N) = (p_1^{k_1} - p_1^{k_1-1}) \dots (p_s^{k_s} - p_s^{k_s-1}).$$

Expanding this product, we will get a sum of 2^s terms, each of the form $\pm d$ for some divisor d of N. Moreover, different terms involve different positive divisor of N, and half of the terms are negative. Thus in the sum $\phi(N) + \sigma(N)$, the negative terms in $\phi(N)$ are canceled by the corresponding divisor in $\sigma(N)$ and the positive terms in $\phi(N)$ contribute twice their value to $\phi(N) + \sigma(N)$. Note that N is among the terms and each positive term is at least $p_1^{k_1-1} p_2^{k_2-1} \dots p_s^{k_s-1}$. Thus

$$\phi(N) + \sigma(N) \geq 2N + (2^{s-1} - 1) p_1^{k_1-1} p_2^{k_2-1} \dots p_s^{k_s-1}.$$

If $\phi(N) + \sigma(N) = 2N + 2$ then we must have $s = 1$ or $s = 2$ and in the latter case $k_1 = k_2 = 1$. Note that if $s = 1$ then $N = p^k$ and $\phi(N) + \sigma(N) = 1 + p + \dots + p^{k-2} + 2p^k$ is never equal to $2N + 2$. Thus

$s = 2$ and $N = pq$ is a product of 2 primes. Conversely, if $N = pq$ is a product of two distinct primes then $\phi(N) + \sigma(N) = 2(N + 1)$. Thus we have the following result:

Proposition. *The equality $\phi(N) + \sigma(N) = 2(N + 1)$ holds if and only if $N = pq$ is a product of two distinct prime numbers.*

Corollary. *N is prosperous if and only if $N = p(p + 4)$ and both p and $p + 4$ are prime numbers.*

Indeed, if N is prosperous then $N = pq$ is a product of two distinct prime numbers by the Proposition and $\phi(N)\sigma(N) = (N - 5)(N + 3) = N^2 - 2N - 15$. We may assume that $p < q$. On the other hand,

$$\phi(N)\sigma(N) = (p - 1)(q - 1)(p + 1)(q + 1) = (p^2 - 1)(q^2 - 1) = N^2 - (p^2 + q^2) + 1.$$

It follows that $p^2 + q^2 - 1 = 2pq + 15$, i.e. $(p - q)^2 = 16$ so $q = p + 4$.
Conversely, it is straightforward from the above discussion that if $N = p(p + 4)$ and both p and $p + 4$ are prime numbers then N is prosperous.

Suppose now that both N and $N - 504$ are prosperous. Thus $N = p(p + 4)$ and $N - 504 = q(q + 4)$, where $p, q, p + 4, q + 4$ are prime numbers. Thus $504 = (p - q)(p + q + 4)$. Note that p, q must be odd so both $p - q$ and $p + q + 4$ are even. There are only six ways of factoring 504 as a product of two even numbers:

$$504 = 2 \cdot 252 = 4 \cdot 126 = 6 \cdot 84 = 12 \cdot 42 = 14 \cdot 36 = 18 \cdot 28.$$

Setting the first factor to be $p - q$ and the second $p + q + 4$ and solving for p, q yields the value of (p, q) to be $(125, 123), (63, 59), (43, 37), (25, 13), (23, 9), (21, 3)$ respectively. The only pair consisting of primes is $(43, 37)$. Thus $N = 43 \cdot 47 = 2021$ and $N - 504 = 37 \cdot 41 = 1517$ and they are both prosperous indeed.

Third Solution. We start by reviewing some basic properties of the functions ϕ and σ

If p is a prime number and $k > 0$ an integer then

$$\phi(p^k) = p^{k-1}(p - 1) \quad \text{and} \quad \sigma(p^k) = 1 + p + \ldots + p^k = (p^{k+1} - 1)/(p - 1). \tag{1}$$

It follows that

$$\phi(p^k)\sigma(p^k) = p^{2k}\left(1 - \frac{1}{p^{k+1}}\right). \tag{2}$$

Indeed, $\phi(p^k)$ is the number of positive integers not exceeding p^k and not divisible by p. Since we have p^{k-1} integers not exceeding p^k and divisible by p (namely $p, 2p, 3p, \ldots, p^{k-1} \cdot p$), thus $\phi(p^k) = p^k - p^{k-1} = p^{k-1}(p - 1)$. Also, the positive divisors of p^k are exactly the numbers $1, p, p^2, \ldots, p^k$, which yields the formula for $\sigma(p^k)$. The equality (2) is an straightforward consequence of (1).

Using the fact that both ϕ and σ are multiplicative, we get the following formula.

If $N = p_1^{k_1} p_2^{k_2} \ldots p_s^{k_s}$ where $p_1 < p_2 < \ldots < p_s$ are distinct prime numbers then

$$\phi(N)\sigma(N) = N^2\left(1 - \frac{1}{p_1^{k_1+1}}\right) \ldots \left(1 - \frac{1}{p_s^{k_s+1}}\right). \tag{3}$$

Suppose now that $N = p_1^{k_1} p_2^{k_2} \ldots p_s^{k_s}$ has the property that

$$\phi(N)\sigma(N) = (N - 5)(N + 3) = N^2 - 2N - 15. \tag{4}$$

Note that N can not be a prime power (i.e. $s > 1$). In fact, if $N = p_1^{k_1}$ then (3) and (4) yield

$$p_1^{2k_1} - p_1^{k_1-1} = p_1^{2k_1} - 2p_1^{k_1} - 15$$

i.e.

$$p_1^{k_1-1}(2p_1 - 1) = -15$$

which is clearly impossible. Note that $\phi(N)$ is even for $N > 2$. Thus the right side of (4) is even, so N is odd. It follows that $N \geq 3 \cdot 5 = 15$. Furthermore, (3) and (4) imply that for any $1 \leq i \leq s$ we have

$$N^2 - 3N \leq N^2 - 2N - 15 = \phi(N)\sigma(N) < N^2 \left(1 - \frac{1}{p_i^{k_i+1}}\right).$$

In other words,

$$\frac{N}{p_i^{k_i}} < 3p_i.$$

Recall that $p_1 < p_2 < \ldots < p_s$. Since $p_2^{k_2} \ldots p_s^{k_s} < 3p_1$, we must have $s = 2$, $k_2 = 1$, and $N = p_1^{k_1} p_2$. Now

$$\phi(N)\sigma(N) = (p_1^{2k_1} - p_1^{k_1-1})(p_2^2 - 1) = N^2 - p_1^{k_1-1}(p_1^{k_1+1} + p_2^2 - 1) \tag{5}$$

so (4) is equivalent to

$$15 = p_1^{k_1-1}(p_1^{k_1+1} + p_2^2 - 2p_1p_2 - 1) = p_1^{k_1-1}(p_1^2(p_1^{k_1-1} - 1) + (p_2 - p_1)^2 - 1). \tag{6}$$

We claim that $k_1 = 1$. Indeed, if $k_1 > 1$ then the right hand side of (6) is at least $3(9 \cdot 2 + 2^2 - 1) = 63$, a contradiction. Thus (6) tells us that $(p_2 - p_1)^2 = 16$ so $p_2 = p_1 + 4$.

To summarize the above discussion, we proved that if N satisfies (4) then $N = p(p+4)$ and both p and $p+4$ are prime numbers. Conversely, if p and $p+4$ are prime and $N = p(p+4)$ then (6) holds and therefore N satisfies (4). Moreover,

$$\phi(N) + \sigma(N) = (p-1)(p+3) + (p+1)(p+5) = 2(p(p+2)+1) = 2(N+1).$$

In other words, N is prosperous.

Our discussion so far can be summarized in the following theorem.

Theorem. *Let N be a positive integer. The following conditions are equivalent.*

1. *N is prosperous.*
2. *$\phi(N)\sigma(N) = (N-5)(N+3)$.*
3. *$N = p(p+4)$, where p and $p+4$ are prime numbers.*

Now we complete the solution in exactly the same way as in the second solution.

Remark. Two prime numbers which differ by 4 are called *cousin primes* (and two prime numbers which differ by 2 are called *twin primes*). It is an open question whether there are infinitely many cousin (or twin) primes.

Remark. I created this problem as a new year puzzler to welcome the year 2021. The name *prosperous* has been coined for the purpose of this problem (as prosperity is one of the new year wishes) and it is by no means an established terminology.
I realized that the second condition in the definition of a prosperous number implies the first one only recently. The original solution I had for the problem is the second solution.

Problem. Prove that N is a product of two twin primes if and only if $\phi(N)\sigma(N) = (N-3)(N+1)$.