Problem 3. A student in a linear algebra class looks at her homework problem. It says: Prove that the product BA is the same for any 3×2 matrix A and 2×3 matrix B such that

$$AB = \begin{bmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ -1 & 1 & \end{bmatrix}.$$

Unfortunately, the entry in the lower right corner of AB is missing. Find the missing entry and solve the homework problem.

Solution. Note that for any invertible 2×2 matrix C we have $AC(C^{-1}B) = AB$. Thus the matrix $(C^{-1}B)(AC) = C^{-1}(BA)C$ is the came for all C. In other words $C^{-1}(BA)C = BA$ for every invertible 2×2 matrix C. This can only happen if BA is a scalar matrix: $BA = dI$ for some number d. Then $dA = A(BA) = (AB)A$. We see that each column of A is an eigenvector of AB with eigenvalue d. Since AB has rank 2, A also has rank 2. Furthermore, the columns of AB are linear combinations of the columns of A, so each column of AB is an eigenvector of AB with eigenvalue d (another way to see this is to note that $dAB = (AB)(AB)$). If $v = [3, 3, -1]^T$ is the first column of AB and x is the missing entry then $ABv = [6, 6, -x]^T = dv = [3d, 3d, -d]^T$. It follows that $d = 2 = x$.

Conversely, suppose that the missing entry is 2. Every column of the matrix AB is then an eigenvector of AB with eigenvalue 2. Since both AB and A have rank 2 and every column of AB is a linear combination of the columns of A, the column spaces of AB and A are the same. It follows that every column of A is an eigenvector of AB with eigenvalue 2. Thus $(AB)A = 2A$, i.e. $A(BA) = 2A$. Since A has rank 2 there is 2×3 matrix D such that $DA = I$. Thus $2I = D(2A) = D(A(BA)) = (DA)BA = BA$, i.e. $BA = 2I$ (another way to show this is to note that $(AB)^2 = 2AB$ and that A has a left inverse D and B has a right inverse E so $BA = DABABE = D(2AB)E = 2I$).

Remark. By M^T we denote the transpose of the matrix M. In particular, if M is a row vector them M^T is a column vector.

Remark. We used the following basic facts from linear algebra:

- If A is an $n \times n$ matrix and $CAC^{-1} = A$ for every invertible $n \times n$ matrix C then A is a scalar matrix: $A = dI$ for some number d, where I is the $n \times n$ identity matrix.
- If A is an $n \times m$ matrix of rank m then there is an $m \times n$ matrix D such that $DA = I$, where I is the $m \times m$ identity matrix.
- If A is an $n \times m$ matrix of rank n then there is an $m \times n$ matrix E such that $AE = I$, where I is the $n \times n$ identity matrix.

Exercise. Prove these three statements.

Remark. The submitted solutions are more ad hoc using the rather simple form of AB.

Ashton notes that

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ -1 & 1 & k \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ -1 & 1 & k \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & -1 & 0 \\ -1 & 1 & k \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & k \end{bmatrix}$$

Thus, if k is the missing entry, then $C \begin{bmatrix} 2 & 0 \\ 0 & k \end{bmatrix} C^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & k \end{bmatrix}$ for any invertible 2×2 matrix C. This implies that $k = 2$.

Both Maxwell and Pluto observe that A has to be of the form

$$A = \begin{bmatrix} a & b \\ a & b \\ c & d \end{bmatrix}$$

with $ad - bc \neq 0$ (which simply means that A has rank 2). Assuming that the missing entry is k, Pluto computes the entries of B in terms of a, b, c, d, k and then computes BA. He then concludes that BA is independent of a, b, c, d iff $k = 2$.

Maxwell establishes that the missing entry is 2 by looking at A, B such that the entry in the upper left corner of A is 0 and then at A', B' such that the entry in the upper right corner of A' is 0. Knowing that the missing entry is 2 and that A has the form above, Maxwell concludes that $ABA = 2A$ and from it he derives that $BA = 2I$.

Problem. Prove that a 3×3 matrix M of rank ≤ 2 has the following property:

the product BA is the same for any 3×2 matrix A and 2×3 matrix B such that $AB = M$

if and only if M has rank 2 and columns of M are eigenvectors of M with the same non-zero eigenvalue (equivalently, $M^2 = dM$ for some non-zero number d). Generalize to higher dimensions.