Problem 2. a) Given three distinct parallel lines on the plane, prove that one can choose one point on each line so that the 3 points are vertices of an equilateral triangle.

b) Given four distinct parallel planes in the space, prove that one can choose one point on each plane so that the 4 points are vertices of a regular tetrahedron.

Solution. a) **Solution 1.** We may introduce a coordinate system on the plane so that the three lines are vertical with equations $x = 0$, $x = s$, and $x = s+t$, where $s > 0$ and $t > 0$. Consider the line $y = ax$ with $a > 0$. It intersects the line $x = s$ at the point $A(a) = (s, as)$ and the line $x = s + t$ at the point $B(a) = (s+t, a(s+t))$. The midpoint $M(a)$ of the segment $A(a)B(a)$ is $M(a) = (s+t/2, a(s+t/2))$. The line L_a perpendicular to $y = ax$ at the midpoint $M(a)$ has equation $y = -1/a(x-(s+t/2))+a(s+t/2)$. The line L_a intersects the line $x = 0$ at the point $C(a) = (0, (a+1/a)(s+t/2))$. The triangle $T(a) = A(a)C(a)B(a)$ is isosceles with $A(a)C(a) = B(a)C(a)$. Note that the segment $A(a)B(a)$ has length $t\sqrt{1 + a^2}$. The height $C(a)M(a)$ has length $h(a) = (s + t/2)\sqrt{1 + a^2}/a$. The tangent of the angle $\angle C(a)A(a)B(a)$ is equal to $2h(a)/A(a)B(a) = (2s + t)/at$. The triangle $T(a)$ is equilateral iff $\angle C(a)A(a)B(a) = \pi/3$, which is equivalent to $(2s+t)/at = \tan(\pi/3) = \sqrt{3}$. Thus when $a = (2s+t)/t\sqrt{3}$ then the triangle $T(a)$ is equilateral.

This solution is constructive: it not only proves the existence but also computes the vertices of the desired triangle. Our second solution will avoid any computations at the expense of not being constructive.

Solution 2. Let our lines be l_1, l_2, l_3. We may assume that the lines l_1 and l_2 are on the same side of the line l_3. Pick a point P on the line l_3 and let $l(\alpha)$ be the line obtained by revolving the line l_3 about P by the angle α counterclockwise, where $0 < \alpha < \pi/2$. The line $l(\alpha)$ intersects the line l_1 at the point $A(\alpha)$ and the line l_2 at the point $B(\alpha)$. Denote by $M(\alpha)$ the midpoint of the segment $A(\alpha)B(\alpha)$ and by t_α the perpendicular bisector of this segment. Let $C(\alpha)$ be the point of intersection of t_α and l_3 (our assumption that the lines l_1 and l_2 are on the same side of l_3 guarantees that $M(\alpha)$ is not on l_3). Let $T(\alpha)$ be the triangle $A(\alpha)B(\alpha)C(\alpha)$. This triangle is isosceles and continuously depends on α. Let $h(\alpha) = C(\alpha)M(\alpha)$. Note that the distance d from $M(\alpha)$ to l_3 does not depend on α (as $M(\alpha)$ is on the line parallel to l_3 which is equally distant from l_1 and l_2). What we conclude from this discussion is that the lengths $b(\alpha) = A(\alpha)B(\alpha)$ and $g(\alpha) = A(\alpha)C(\alpha) = \sqrt{4h(\alpha)^2 + b(\alpha)^2}/2$ are continuous functions of α. When α approaches 0 then $b(\alpha)$ tends to infinity and $h(\alpha)$ tends to d (as the line $l(\alpha)$ is almost parallel to l_3). It follows that $g(\alpha) < b(\alpha)$ for α large enough. When α approaches $\pi/2$, we have $b(\alpha)$ approaches the distance between l_1 and l_2 and $h(\alpha)$ tends to infinity (as the line $l(\alpha)$ is almost perpendicular to l_3). Thus $g(\alpha) > b(\alpha)$ for α small enough. It follows from continuity that there is an angle α for which $g(\alpha) = b(\alpha)$. For this angle the triangle $T(\alpha)$ is equilateral.

Solution 3 (after Pluto Wang). Pick a line l perpendicular to the given three lines which intersects them at points P, Q, R respectively. We may assume that Q is between P and R and $PQ \leq QR$. We will denote by l_P, l_Q, l_R the line which passes through P, Q, R respectively and is perpendicular to l (so these are the three given parallel lines). On the line l_P there is a point X such that $QX = QR$ (there are at most two such points; pick one of them). For any $a > 0$ there is unique point $A = A(a)$ on the ray $\overrightarrow{PX}$ such that $PA = a + PX$. Since $QA \geq QX = QR$, there is unique point $B = B(a)$ on the line l_R such that $QB = QA$ and B is on the same side of the line l as A. The triangle AQB is isosceles. When a changes from 0 to infinity (i.e. the point A moves along the ray $\overrightarrow{PX}$), the angle $\angle AQB$ changes continuously from $\angle XQR$ to 0. Since $\angle XQR$ is obtuse (hence bigger than $\pi/3$) there is a value of a for which the angle $\angle AQB$ is $\pi/3$ (by the intermediate value theorem). For this value of a the triangle AQB is equilateral.
Remark. Pluto actually finds an explicit formula for the angle $\angle AQB$ as a function of a to see that the angle changes continuously with a.

b) Our solution to part b) will follow the line of reasoning of the second solution to part a). Let our planes be p_1, p_2, p_3, p_4 and let the planes p_1, p_2, p_3 be all on the same side of the plane p_4. Fix a line l on the plane p_4 and let $p(\alpha)$ be the plane obtained by revolving the plane p_4 about the line l by the angle α counterclockwise, where $0 < \alpha \leq \pi/2$. The plane $p(\alpha)$ intersects the planes p_1, p_2, p_3 at the lines $l_1(\alpha), l_2(\alpha), l_3(\alpha)$ respectively, which are mutually parallel. By part a), there is an equilateral triangle $T(\alpha)$ whose vertices $A_1(\alpha), A_2(\alpha), A_3(\alpha)$ are on the lines $l_1(\alpha), l_2(\alpha), l_3(\alpha)$ respectively. The side $b(\alpha)$ of this triangle is a continuous function of α (this claim, while intuitively clear, requires a more careful analysis of part a); it follows rather easily from our solution 1). Denote by $M(\alpha)$ the center of the equilateral triangle $T(\alpha)$ and by t_α the line perpendicular to the plane $p(\alpha)$ at the point $M(\alpha)$ (note that $M(\alpha)$ does not belong to p_4). Let $C(\alpha)$ be the point of intersection of t_α and the plane p_4. Let $D(\alpha)$ be the tetrahedron $A_1(\alpha)A_2(\alpha)A_3(\alpha)C(\alpha)$. Let $h(\alpha) = C(\alpha)M(\alpha)$, so h is a continuous function of α. Note that the distance d from $M(\alpha)$ to p_4 does not depend on α (we only need that this distance is bounded, and it is clear that it does not exceed the largest of the distances between p_4 and the planes p_1, p_2, p_3). Recall that $A_1(\alpha)M(\alpha) = 2b(\alpha)/3$. Let

$$g(\alpha) = A_1(\alpha)C(\alpha) = \sqrt{h(\alpha)^2 + A_1(\alpha)M(\alpha)^2} = \sqrt{9h(\alpha)^2 + 4b(\alpha)^2}/9.$$

The tetrahedron $D(\alpha)$ is regular if and only if $g(\alpha) = b(\alpha)$. When α approaches 0 then $b(\alpha)$ tends to infinity and $h(\alpha)$ tends to d (as the plane $p(\alpha)$ is almost parallel to p_4). It follows that $g(\alpha) < b(\alpha)$ for α large enough. When α approaches $\pi/2$ then $b(\alpha)$ approaches $b(\pi/2)$ and $h(\alpha)$ tends to infinity (as the plane $p(\alpha)$ is almost perpendicular to p_4). Thus $g(\alpha) > b(\alpha)$ for α small enough. Since both b and g are continuous functions of α, there is an angle α for which $g(\alpha) = b(\alpha)$. For this angle the tetrahedron $D(\alpha)$ is regular.

Remark. The solution provided by Ashton Keith is actually explicit; it introduces appropriate coordinates and computes the angle α in terms of the distances between the planes in a way similar to our first solution to part a). We do not provide details here.

Problem. Generalize this problem to higher dimensions.

Problem. Consider a triangle T. In part a) of the problem, is it possible to choose one point on each line so the resulting triangle is similar to T? Study analogous question in higher dimensions.