Problem 1. In certain village there are 333 voters. Each voter belongs to one of two parties: knights or knaves. Knaves sometimes lie and sometimes tell the truth, while knights always tell the truth. Every voter knows the party affiliation of all voters and there are more knights than knaves. You, the mathematician, arrive to the village with the task to meet a knight. You can ask each voter the following question abut any voter: is she/he a knight? Find a knight by asking as few questions as you can.

Solution. The problem was formulated in an open-ended way not only to allow flexibility in potential solutions but also because we do not know what the most efficient solution is. The most efficient solution we know has been submitted by Ashton Keith.

We start with the following simple, but key, observations which will be used in all solutions below. Suppose that voter A is asked about voter B (recall the question: is B a knight?). Then

(1) if A answers *no* then at least one of A and B is a knave.

(2) if A answers *yes* and B is a knave then A is a knave too.

Solution 1 (simple but very inefficient). Pick one voter V and ask every other voter whether V is a knight. Suppose we get y answers *yes* and n answers *no*. Thus $y + n = 332$.

Suppose $y \geq n$, so $y \geq 166$. If V was a knave then all who said *yes* would be knaves and we would have at least $166 + 1 = 167$ knaves, which is impossible. Thus V is a knight in this case.

Suppose now $n > y$, so $n \geq 167$. If V was a knight then all who said *no* would lie and we would have at lest 167 knaves, which is not possible. Thus V is a knave in this case. Moreover all knights are among those who said *no*. Now we can focus on those who said *no* and repeat the procedure until we find a knight.

We will not make any effort to get exact estimate of number of steps this procedure requires in the worst case. It should be clear though that this number will be huge (around $2 + 4 + \ldots + 332 = 166 \cdot 167$). The main value of this solution is in showing that the problem can be solved.

Solution 2 (again simple but very inefficient). This solution is based on the following consequence of (2): if we have voters $A_1, \ldots, A_{167}$ and for every $1 \leq i < 167$ voter A_i says *yes* when asked about voter A_{i+1} then voter A_{167} is a knight (as otherwise all voters A_i would be knaves). Thus we keep asking questions until we get such a chain of 167 voters. This must happen at some point, as after asking every voter about every other voter, any 167 knights in any order will form such a chain. However, it should be clear that this procedure is very inefficient.

The next two solutions will be much more efficient. We will discuss them for arbitrary number m of voters as the solutions will be recursive in nature.

Solution 3. We will show that we can find a knight by asking at most $m - 2$ questions. This will be done by induction on m (as we will reduce the process of solving the problem for m voters to the same process for $m - 2$ voters). We number the voters from 1 to m. Our procedure starts with asking the first voter about the second, then the second about third, etc. until we get *no* as the answer. If the first $\left\lceil \dfrac{m}{2} \right\rceil - 1$ answers are *yes* then voter number $\left\lceil \dfrac{m}{2} \right\rceil$ is a knight and we are done (see solution 2).

If we get a *no* when asking k-th voter (where $k < \lceil \frac{m}{2} \rceil - 1$) , then we remove voters 1 and $k + 1$ and continue the procedure (starting by asking voter k about voter $k + 2$). It is easy to see that we are just applying our procedure to voters $2, \ldots, k, k + 2, \ldots, m$. Note that at least one of the removed voters $1, k + 1$ is a knave (if $k + 1$ is a knight then all voters $1, \ldots, k$ lied, hence are knaves). Thus among the $m - 2$ voters $2, \ldots, k, k + 2, \ldots, m$ there are more knights than knaves. By induction, our process will find a knight among the voters $2, \ldots, k, k + 2, \ldots, m$ after no more that $(m - 2) - 2 = m - 4$ questions. We also need to count the question to voter 1 about voter 2 and (when $k > 1$) the question to voter k about voter $k + 1$. Thus we find a knight by asking no more than $m - 4 + 2 = m - 2$ questions.

Solution 4 (a slightly improved and edited version of the solution provided by Ashton Keith). We number the voters from 1 to m. Let $f(m)$ be the smallest number of questions required to solve the problem with m voters. Thus $f(1) = f(2) = 0$, since all voters must be knights when $m \leq 2$. Note that $f(3) = 1$. Indeed, we ask voter 1 about voter 2. If the answer is *no*, voter 3 is a knight. If the answer is *yes*, voter 2 is a knight.

Note that $f(2k) \le f(2k-1)$ for any k. Indeed, given $2k$ voters, removing any voter will leave us with $2k-1$ voters among which there is more knights than knaves (it is also not hard to see that $f(2k) \le f(2k+1)$; just add a dummy knave). This observation allows us to focus on odd numbers m.

Let $m = 2k+1$. We start by asking voter i about voter $i+1$ for every odd number $i < 2k$ (so we ask k questions). Suppose we get n answers *no* and y answers *yes*, so $n + y = k$. Note that if voter i answered *no* then at least one of i,i+1 is a knave. Thus, if $y = 0$ then we have at lest k knaves among voters $1, ..., 2k$. This means that voter $2k+1$ is a knight (as there are at most k knaves) and we are done. Suppose now that $y > 0$. Let T be the set of even numbered voters about whom the answer was *yes*. Recall that if voter i answers *yes* about $i+1$ and $i+1$ is a knave then i is a knave too. Thus if T has f knaves and t knights then $f + t = y$ and we have at least $n + 2f$ knaves among voters $1, ..., 2k$. Thus $n + 2f \le k = n + y = n + f + t$. Thus $t \ge f$. If y is odd then $t > f$ (as $t = f$ is not possible), so T contains more knights than knaves and we reduce our problem to finding a knight among voters in T. If y is even consider the set T' obtained by adding to T the voter $2k+1$. Note that if $t > f$ then $t > f+1$ (as $t+f$ is even) so the set T' has more knights than knaves. If $t = f$ then $y = t + f = 2f$ and we have at lest $n + 2f = k$ knaves among $1, 2, ..., 2k$. This means that voter $2k+1$ is a knight and again T' contains more knights than knaves. Thus we reduced the problem to finding a knight among voters in T'.

The above discussion reduces the problem to finding a knight among the voters in the set T or T'. However, we do not know the size of this set. We only know that it does not exceed $k+1$. Since we do not know that the function f is increasing, let us define a new function g by setting

$$g(m) = \max\{f(i) : 1 \le i \le m\}.$$

Then g is nondecreasing, $g(2k) = g(2k-1)$ for all k, and $f(m) \le g(m)$ for all m. We can now phrase our analysis above as a simple inequality: $g(2k+1) \le k + g(k+1)$ for every k. We now need to extract from this inequality a good upper bound for $g(2k+1)$.

We can define a function h inductively as follows: $h(0) = h(1) = 0$ and

$$h(m) = \begin{cases} h(m-1) & \text{if } m \text{ is even} \\ \frac{m-1}{2} + h\left(\frac{m+1}{2}\right) & \text{if } m \text{ is odd.} \end{cases}$$

Then a straightforward induction shows that $g(m) \le h(m)$ for all m. Is there a more explicit formula for h? After computing the values $h(m)$ for the first several m (say for $m \le 20$) it is not hard to realize that $h(m) = m - s(m)$ for every odd m, where $s(m)$ is the number of digits 1 in the binary expansion of m. Once we realize this formula, proving it is not hard. Indeed, note that $s(k) = s(2k)$ and $s(2k+1) = 1 + s(2k) = 1 + s(k)$ for every k. We use induction. Assuming that $h(i) = i - s(i)$ for all odd $i < m = 2k+1$ we have

$$h(m) = k + h(k+1) = k + h(k) = k + k - s(k) = 2k - s(k) = 2k + 1 - (1 + s(k)) = m - s(m)$$

if k is odd, and

$$h(m) = k + h(k+1) = k + k + 1 - s(k+1) = m - (1 + s(k)) = m - s(m)$$

if k is even.

Putting all the above discussion together we see that

$$f(2k) \le f(2k-1) \le h(2k-1) = 2k - 1 - s(2k-1)$$

for every positive integer k. In particular, $s(333) = 5$ and we can find a knight by asking at most $333 - 5 = 328$ questions.

Challenge. Can one do better? In other words, is it true that $f(m) = h(m)$ for all m?

We do not know the answer to this question.

Note that one can imagine two types of procedures to solve our problem. One type allows to decide next question based on the answers received in the previous questions. Both solutions 3 and 4 are of this type. A second, more restrictive type requires to decide the order of questions ahead of time. Solutions 1 and 2 can be easily adjusted to be of the second type.

Problem. What is the most efficient procedure of the second type?

Finally, one can ask the following variation of our problem:

Problem. Find the most efficient procedure to determine party affiliation of all voters.

Note that if we know a knight, then we can ask him about every other voter, so the last problem can be solved by asking no more than $f(m) + m - 1$ questions. But one can do better. In fact, we do know the most efficient procedure for this problem. For example, when $m = 333$ we can find party affiliation of all voters by asking no more than 498 questions and this is best possible (our solution 3 should lead to such a procedure).