Problem 3. Recall that a *chord* of a circle is a straight line segment whose endpoints both lie on the circle.

Several chords are drawn in a circle of radius 1 so that any diameter of the circle intersects at most k of the chords. Prove that the sum of the lengths of all the chords drawn is less than $k\pi$.

Solution. Let us start by setting some terminology and making a few simple observations. A chord $\overline{AB}$ in a circle divides the circle into two arcs with ends A, B. One of these arcs, called the *minor arc* associated with the chord, is contained in a semicircle. The other arc contains a semicircle and it is called the *major arc* associated with the chord. When the cord is a diameter then this terminology is ambiguous. For our purpose, we choose one of the two semicircles associated with a diameter and call it a minor arc.

It is intuitively clear that the length of a chord is smaller that the length of its minor arc. More precisely, if a minor arc C in a unit circle with center O has ends A, B and α is the central angle $\angle AOB$ expressed in radians, then the length of C is α and the length of the chord $\overline{AB}$ is $2\sin(\alpha/2)$. Thus our claim follows from the well known inequality $\sin x < x$, true for all $x > 0$.

Any arc C of a circle has its *reflected* arc C^*, which is the reflection of C over the center of the circle. It is a straightforward observation that if C is a minor arc associated with a chord then the following three statements are equivalent:

- a diameter d intersects the chord;
- a diameter d has one end on the arc C;
- a diameter d has one end on the arc C^*.

Moreover, if C is a minor arc then C and C^* have no interior points in common.

Now we can get to the solutions of our problem.

First (our original) solution. Suppose that the sum of the lengths of the drawn chords is at least $k\pi$. We will see that this assumption leads us to a contradiction. Let $\mathcal{F}$ be the collection of minor arcs associated to the drawn chords and let $\mathcal{F}^*$ be the set consisting of all reflected arcs from the set $\mathcal{F}$. Since an arc is longer that its associated chord and any arc and its reflected arc have the same length, we conclude that the sum of the lengths of all the arcs in $\mathcal{F} \cup \mathcal{F}^*$ is more that $2k\pi$. We claim that this implies that there is a point P which belongs to the interiors of at least $k+1$ of the arcs in $\mathcal{F} \cup \mathcal{F}^*$. Suppose the claim is true. Then, according to our remarks above, the diameter starting at P intersects $k+1$ of the chords drawn, a contradiction.

It remains to justify the claim. Consider all the ends of the arcs in the set $\mathcal{F} \cup \mathcal{F}^*$. Starting at one of these ends and going around the circle clockwise number the ends $E_1, E_2, \ldots, E_m$ in the order in which you encounter them. Consider the little arcs E_1E_2, E_2E_3,..., $E_{m-1}E_m$, E_mE_1. Any arc C in the set $\mathcal{F} \cup \mathcal{F}^*$ is a union of some of these little arcs. If each little arc were contained in at most k of the arcs in the set $\mathcal{F} \cup \mathcal{F}^*$, then the sum of lengths of all the arcs in $\mathcal{F} \cup \mathcal{F}^*$ would not exceed k times the sum of the lengths of all the little arcs, which is exactly $2k\pi$, a contradiction. Thus one of the little arcs must be contained in at least $k+1$ different arcs in $\mathcal{F} \cup \mathcal{F}^*$, which implies our claim.

Second solution (due to Ashton Keith). Fix a point X on our circle and choose one of the semicircles on the diameter from X, call it D. For any number α in the interval $[0, \pi)$ there is unique point Y on D such that the central angle $\angle XOY = \alpha$. We identify α with the diameter from Y. This establishes a bijection between diameters of our circle and numbers in $[0, \pi)$.

For any minor arc C of the circle define a function $f_C : [0, \pi) \longrightarrow \mathbb{R}$ as follows:

$$f_C(\alpha) = \begin{cases} 1 & \text{if the diameter corresponding to } \alpha \text{ intersects } C \\ 0 & \text{otherwise.} \end{cases}$$

Note that $f_C(\alpha) = 1$ if and only if the diameter corresponding to α intersects the chord associated to C (this requires C to be a minor arc).

We leave it to the reader to show that

$$\int_0^\pi f_C(\alpha)\, d\alpha = \text{length of } C$$

(hint: split C into two arcs, one contained in D and the other disjoint from D; note that the formula does not work for major arcs).

Consider now the collection $\mathcal{F}$ of all minor arcs associated to the drawn chords. Let

$$g(\alpha) = \sum_{C \in \mathcal{F}} f_C(\alpha).$$

Then

$$\int_0^{\pi} g(\alpha)\ d\alpha = \text{sum of the lengths of arcs in } \mathcal{F}.$$

On the other hand, our assumption says that for every α we have $f_C(\alpha) \neq 0$ for at most k arcs C in $\mathcal{F}$. It follows that $g(\alpha) \leq k$ for every α. Consequently,

$$\text{sum of the lengths of arcs in } \mathcal{F} = \int_0^{\pi} g(\alpha)\ d\alpha \leq \int_0^{\pi} k\ d\alpha = k\pi.$$

Since the sum of the lengths of all the drawn chords is smaller that the sum of the lengths of all the arcs in $\mathcal{F}$, the result follows.

Remark. The last part of our first solution is a special case of a more general fact, which may be useful in many other problems.

Suppose that we have a family $\mathcal{M}$ of subsets of a set X which is closed under taking intersection, union, and difference of any two sets in $\mathcal{M}$. Suppose that we have a function m which assigns to any set A in $\mathcal{M}$ a non-negative real number $m(A)$, called the measure of A. We assume that for any two disjoint subsets A, B in $\mathcal{M}$ the equality $m(A \cup B) = m(A) + m(B)$ holds (a typical example of a function m would be length, area, or volume).

Suppose now that $\mathcal{F}$ is a finite subset of $\mathcal{M}$ and that T is the union of all subsets in $\mathcal{F}$.

Theorem. If

$$\sum_{A \in \mathcal{F}} m(A) > km(T),$$

then there is an element which belongs to at lest $k+1$ of the sets in $\mathcal{F}$.

We leave it as an exercise to prove this result. Hint: Suppose that $\mathcal{F} = \{A_1, \ldots, A_n\}$. We set $A^1 = A$ and $A^{-1} = T - A$ (the complement of A in T). For any sequence $e = (e_1, \ldots, e_n)$ of ± 1 consider the set

$$B_e = A_1^{e_1} \cap A_2^{e_2} \cap \ldots \cap A_n^{e_n}.$$

The sets B_e should play the role of the little arcs in the last part of our first solution.

Note also, that if we had a notion of integral for some functions on the set X, including functions of the form

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

for every $A \in \mathcal{M}$, obeying the usual rules and such that

$$\int f_A = m(A)$$

then we could use the method of the second solution to prove the theorem.