Problem 7. A female soccer club has 23 players. The strength of each player is a positive integer assigned to the player based on her performance in the last 3 seasons. It turns out that leaving any player aside, the remaining 22 players can be divided into two teams of 11 so that the sum of strengths of all players in each team is the same. Prove that all players have the same strength.

Solution. We will solve a slightly more general problem where the strength of a player is an arbitrary integer (rather than just a positive integer). Let s_i denote the strength of the i-th player. Suppose that the strengths s_i satisfy the condition of the problem. Our solution is based on the following rather straightforward observations:

(1) All the strengths s_i have the same parity.

Indeed, let s be the sum of all the strengths. Then for each i the number $s - s_i = 2k_i$ is even, where k_i is the sum of the strengths of all players in each of the 2 teams we can build according to the problem using all the players except the i-th player. It follows that each s_i has the same parity as s.

(2) Adding (or subtracting) and integer m to each s_i yields strengths which still satisfy the conditions of the problem.

(3) If each s_i is divisible by some integer k then replacing s_i by s_i/k for every i yields strengths which still satisfy the conditions of the problem.

Now consider our strengths s_i. Subtract s_1 from all of them: $s_i^* = s_i - s_1$. According to (2), the new strengths s_i^* satisfy the condition of the problem and $s_1^* = 0$. According to (1) all s_i^* are even. Now divide them by 2. By (3), the strengths $s_i^*/2$ satisfy the conditions of the problem, so they are all even by (1) and the fact that $s_1^*/2 = 0$. Now repeat this argument again and again. We see that for every $k > 0$ the numbers $s_i^*/2^k$ are integers. Note that the only integer divisible by arbitrary high powers of 2 is 0. Thus all s_i^* must be equal to 0. Consequently, the original s_i are all equal. This completes our solution.

Remark. Suppose that we allow the strengths to be rational numbers. Let d be the least common multiple of all the denominators of each strength. Multiplying each strength by d yields strengths which are integers and still satisfy the condition of the problem. Hence they are all equal, so the original strengths are equal as well.

What if the strengths are arbitrary real numbers? One way to see that they still have to be equal is to use some basic linear algebra. The conditions on the strengths s_i imposed by the problem can be phrased as follows: for every i there exist numbers $a_{i,1}, a_{i,2}, \ldots, a_{i,23}$ such that $a_{i,i} = 0$, $a_{i,j} = 1$ for eleven indexes j, $a_{i,j} = -1$ for the remaining 11 indexes j, and

$$a_{i,1}s_1 + a_{i,2}s_2 + ... + a_{i,23}s_{23} = 0.$$

Let A be the 23×23 matrix $(a_{i,j})$. Thus A has 0's on the diagonal and each row has 11 entries equal to 1 and 11 entries equal to -1. The numbers $s_1, \ldots, s_{23}$ are strengths satisfying the conditions of the problem if and only if $As = 0$, where $s = (s_1, \ldots, s_{23})$. We proved that over the field of rational numbers any solution to this homogeneous system of 23 linear equations with 23 unknowns satisfies $s_1 = \ldots = s_{23}$. This means that the solution space of this system is one dimensional, so the rank of the matrix A is 22. Well, when we consider this system over the real numbers the rank of A is still 22. So the solution space is still of dimension one, i.e. real numbers $s_1, \ldots, s_{23}$ form a solution iff $s_1 = s_2 = \ldots = s_{23}$.

Exercise. Let n be even and let A be an $n \times n$ matrix of integers in which every diagonal entry is even and every off-diagonal entry is odd. Prove that the rank of A is n (one way is to appropriately modify the method we used to solve our problem). Also, show that this result easily implies the conclusion of our problem.