Problem 6. Let $f(x)$ be a polynomial with real coefficients such that $f(x) - 2f'(x) + f''(x) > 0$ for all x. Prove that $f(x) > 0$ for all x.

Solution. We will prove first the following result:

Proposition. Let $g(x)$ be a polynomial with real coefficients such that $g(x) - g'(x) > 0$ for all x. Then $g(x) > 0$ for all x.

Proof: Let g be of degree n with leading coefficient a, so $g(x) = ax^n +$ terms of lower degree. If $n = 0$ (i.e. g is constant) the result is clear. We assume now that $n > 0$. Then $h(x) = g(x) - g'(x)$ is also a polynomial of degree n with leading cooefficient a. Since h is always positive, we must have n even and $a > 0$. This means that $\lim_{x\to-\infty} g(x) = +\infty = \lim_{x\to+\infty} g(x)$. Thus g assumes its smallest value at some $a \in \mathbb{R}$. It follows that $g'(a) = 0$ and $g(a) = g(a) - g'(a) > 0$. Since the smallest value of g is positive, the proposition follows.

Second method. We give a different proof of the proposition, based on the following nice observation: if $G(x) = g(x)e^{-x}$ then $G'(x) = (g'(x) - g(x))e^{-x}$. When g is a polynomial is in the proposition then we see that $G'(x) < 0$ for all x. This means that the function G is decreasing. Since g is a polynomial, we have $\lim_{x\to+\infty} G(x) = 0$. These two facts together tell us that $G(x) > 0$ for all x, which is equivalent to $g(x) > 0$ for all x.

Remark. Note that this method allows to replace the assumption that g is a polynomial by a much weaker assumption that $\lim_{x\to+\infty} g(x)e^{-x} = 0$.

We can now solve the problem. Let $g(x) = f(x) - f'(x)$. Then $g(x) - g'(x) = f(x) - 2f'(x) + f''(x) > 0$ for all x. By the proposition we conclude that $g(x) = f(x) - f'(x) > 0$ for all x. Using the proposition again, we see that $f(x) > 0$ for all x.

Exercise. Solve the problem with the assumption that f is a polynomial replaced by the assumption that f is twice differentiable and $\lim_{x\to+\infty} f'(x)e^{-x} = 0$.

Exercise. Let $f(x)$ be a polynomial with real coefficients such that $f(x) + 3f'(x) + 3f''(x) + f'''(x) > 0$ for all x. Prove that $f(x) > 0$ for all x.