Problem 5. Recall that $\lfloor a \rfloor$ denotes the floor of a, i.e. the largest integer smaller or equal than a. What is the smallest possible value of $\left\lfloor \frac{1}{x_1} \right\rfloor + \left\lfloor \frac{1}{x_2} \right\rfloor + \ldots + \left\lfloor \frac{1}{x_n} \right\rfloor$, where $x_1, x_2, \ldots, x_n$ are positive real numbers such that $x_1 + \ldots + x_n = 1$?

Solution. Our solution is based on the following well known inequality, true for any positive real numbers $x_1, \ldots, x_n$:

$$(x_1 + \ldots + x_n)(\frac{1}{x_1} + \ldots + \frac{1}{x_n}) \geq n^2 \tag{1}$$

(for those familiar with inequalities, (1) is equivalent to the AMHM inequality, i.e. inequality between the arithmetic and harmonic means). For convenience, we provide a proof of this inequality at the end of our solution.

When $x_1 + \ldots + x_n = 1$, we get the inequality $\frac{1}{x_1} + \ldots + \frac{1}{x_n} \geq n^2$. Now note that $\lfloor x \rfloor > x - 1$ for any x. It follows that

$$\left\lfloor \frac{1}{x_1} \right\rfloor + \left\lfloor \frac{1}{x_2} \right\rfloor + \ldots + \left\lfloor \frac{1}{x_n} \right\rfloor > \left(\frac{1}{x_1} - 1 \right) + \ldots + \left(\frac{1}{x_n} - 1 \right) \geq n^2 - n$$

Since $\left\lfloor \frac{1}{x_1} \right\rfloor + \left\lfloor \frac{1}{x_2} \right\rfloor + \ldots + \left\lfloor \frac{1}{x_n} \right\rfloor$ is an integer larger than $n^2 - n$, it is at least $n^2 - n + 1$, i.e.

$$\left\lfloor \frac{1}{x_1} \right\rfloor + \left\lfloor \frac{1}{x_2} \right\rfloor + \ldots + \left\lfloor \frac{1}{x_n} \right\rfloor \geq n^2 - n + 1.$$

Taking $x_1 = x_2 = \ldots = x_{n-1} = 1/n + \epsilon$ and $x_n = 1/n - (n-1)\epsilon$, where $\epsilon > 0$ is very small, we have $\lfloor 1/x_i \rfloor = n - 1$ and $\lfloor 1/x_n \rfloor = n$, so

$$\left\lfloor \frac{1}{x_1} \right\rfloor + \left\lfloor \frac{1}{x_2} \right\rfloor + \ldots + \left\lfloor \frac{1}{x_n} \right\rfloor = (n-1)^2 + n = n^2 - n + 1.$$

As a matter of fact, any $\epsilon < 1/(n^3 - n)$ will work. More explicitely, one can take

$$x_1 = \ldots = x_{n-1} = \frac{2n-1}{(2n+1)(n-1)} \quad \text{and} \quad x_n = \frac{2}{2n+1}$$

(so $\epsilon = 1/(n(n-1)(2n+1))$).

We see that the smallest value of $\left\lfloor \frac{1}{x_1} \right\rfloor + \left\lfloor \frac{1}{x_2} \right\rfloor + \ldots + \left\lfloor \frac{1}{x_n} \right\rfloor$ is $n^2 - n + 1$.

Proof of the inequality (1). Our starting point is the inequality $a + b \geq 2\sqrt{ab}$, which holds for any positive a, b. Indeed, this inequality is equivalent to $(\sqrt{a} - \sqrt{b})^2 \geq 0$, which is clearly true. From this we make two conclusions:

$$\frac{A}{x} + Bx \geq 2\sqrt{AB} \quad \text{for any positive} \quad A, B, x \tag{2}$$

and

$$a + 1/a \geq 2 \quad \text{for any positive} \quad a. \tag{3}$$

One way to prove (1) is to proceed by induction on n. Indeed, there is nothing to prove when $n = 1$. Assuming that for $A = x_1 + \ldots + x_n$, $B = 1/x_1 + \ldots + 1/x_n$ we have $AB \geq n^2$ we conclude that

$$(A + x_{n+1})(B + \frac{1}{x_{n+1}}) = AB + 1 + \frac{A}{x_{n+1}} + Bx_{n+1} \geq AB + 1 + 2\sqrt{AB} = (\sqrt{AB} + 1)^2 \geq (n+1)^2.$$

(we used inequality (2)). Thus the induction step works and (1) holds for all n.

A different argument goes as follows: expand $(x_1 + \ldots + x_n)(1/x_1 + \ldots 1/x_n)$ to get a sum of n^2 terms, n of which are equal 1 (as $x_i(1/x_i) = 1$) and the remaining terms split into $(n^2 - n)/2$ pairs of the form $x_i/x_j + x_j/x_i$. By (3), each such sum is at least 2, so we get

$$(x_1 + \ldots + x_n)(\frac{1}{x_1} + \ldots + \frac{1}{x_n}) \geq n + \frac{n^2 - n}{2} \cdot 2 = n^2$$

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Exercise. Show that we have equality in (1) if and only if $x_1 = \ldots = x_n$.

Second solution (after Yuqiao Huang):
Since the values of $f(x_1, \ldots, x_n) = \left\lfloor \frac{1}{x_1} \right\rfloor + \left\lfloor \frac{1}{x_2} \right\rfloor + \ldots + \left\lfloor \frac{1}{x_n} \right\rfloor$ are positive integers, the function f on the set of positive real numbers with sum 1 attains its smallest value M at some point $(x_1, ..., x_n)$. It is easy to see that $M \leq n^2 - n + 1$ Indeed, there is $\epsilon > 0$ such that $1/n < 1/n + \epsilon < 1/(n-1)$ and $1/(n+1) < 1/n - (n-1)\epsilon < 1/n$. Taking $x_1 = \ldots = x_{n-1} = 1/n + \epsilon$ and $x_n = 1/n - (n-1)\epsilon$ we get $f(x_1, \ldots, x_n) = n^2 - n + 1$.

The key idea of the solution is to manipulate $x_1, ..., x_n$ to get a different point $y_1, ..., y_n$ where the minimum is attained which has the property that each $\lfloor 1/y_i \rfloor$ is either n or $n-1$. It follows then that $M = k(n-1) + (n-k)n = n^2 - k$ for some integer $k \leq n$. Combined with the inequality $M \leq n^2 - (n-1)$ we easily see that $k = n-1$ and $M = n^2 - n + 1$ (note that $k = n$ is not possible, as this would mean $\lfloor 1/y_i \rfloor = n - 1$ for all n, which implies that $1/y_i < n$, i.e. $y_i > 1/n$ for all i, contradicting the equality $y_1 + \ldots + y_n = 1$.

It remains to prove the existence of $y_1, \ldots, y_n$. Our proof is based on the following key observation:

Lemma. Let x, y be real numbers in $(0,1)$ such that $k = \left\lfloor \frac{1}{x} \right\rfloor < \left\lfloor \frac{1}{y} \right\rfloor = m$. This means that $\frac{1}{k+1} < x \leq \frac{1}{k}$, $\frac{1}{m+1} < y \leq \frac{1}{m}$, and $k + 1 \leq m$. Let $d(x, y) = d = \left(x - \frac{1}{k+1}\right) + \left(\frac{1}{m} - y\right)$. Then $\left\lfloor \frac{1}{x-d} \right\rfloor = k+1$ and $\left\lfloor \frac{1}{y+d} \right\rfloor \leq m - 1$.

Proof of Lemma. Note that $\frac{1}{k+2} < \frac{1}{k+1} - \frac{1}{m} + \frac{1}{m+1}$, since $\frac{1}{k+1} - \frac{1}{k+2} = \frac{1}{(k+1)(k+2)} \geq \frac{1}{m(m+1)} = \frac{1}{m} - \frac{1}{m+1}$. Thus

$$\frac{1}{k+2} < \frac{1}{k+1} - \frac{1}{m} + \frac{1}{m+1} < \frac{1}{k+1} - \frac{1}{m} + y = x - d \leq \frac{1}{k+1} - \frac{1}{m} + \frac{1}{m} = \frac{1}{k+1}.$$

This means that $k + 2 > \frac{1}{x-d} \geq k + 1$, i.e. $\left\lfloor \frac{1}{x-d} \right\rfloor = k + 1$. Also, $y + d = x - \frac{1}{k+1} + \frac{1}{m} > \frac{1}{m}$, so $\left\lfloor \frac{1}{y+d} \right\rfloor \leq m - 1$.

Consider now the set

$$\mathcal{F} = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \text{ are positive numbers}, \quad x_1 + \ldots + x_n = 1, \quad \text{and} \quad f(x_1, \ldots, x_n) = M\}$$

of all points where f attains its smallest value M. This set has the following properties:

1. If $(x_1, \ldots, x_n) \in \mathcal{F}$ then $\left\lfloor \frac{1}{x_i} \right\rfloor \geq n$ for some i.
2. If $(x_1, \ldots, x_n) \in \mathcal{F}$ then $\left\lfloor \frac{1}{x_i} \right\rfloor \leq n - 1$ for some i.
3. If $(x_1, \ldots, x_n) \in \mathcal{F}$, $\left\lfloor \frac{1}{x_i} \right\rfloor \leq n - 1$, $\left\lfloor \frac{1}{x_j} \right\rfloor \geq n$ for some i, j, and $d = d(x_i, x_j)$, then replacing x_i, x_j with $x_i - d$ and $x_j + d$ and keeping all the other x_k's intact yields another element of $\mathcal{F}$.

Indeed, if $\left\lfloor \frac{1}{x_i} \right\rfloor < n$ for all i, then $\frac{1}{x_i} < n$ for all i, i.e. $x_i > 1/n$ for all i, which contradicts the equality $x_1 + \ldots + x_n = 1$. This proves 1.

Similarly, if $\left\lfloor \frac{1}{x_i} \right\rfloor \geq n$ for all i, then $\frac{1}{x_i} \geq n$ for all i, i.e. $x_i \leq 1/n$ for all i. The equality $x_1 + \ldots + x_n = 1$ implies that $x_i = 1/n$ for all i. However $(1/n, 1/n, \ldots, 1/n)$ does not belong to $\mathcal{F}$, since $f(1/n, \ldots, 1/n) = n^2$ and $M \leq n^2 - n + 1$. The contradiction proves 2.

In order to justify 3. we use the Lemma. It tells us that $\left\lfloor \frac{1}{x_i - d} \right\rfloor = \left\lfloor \frac{1}{x_i} \right\rfloor + 1$ and $\left\lfloor \frac{1}{x_j + d} \right\rfloor \leq \left\lfloor \frac{1}{x_j} \right\rfloor - 1$. Setting $z_i = x_i - d$, $z_j = x_j + d$, $z_k = x_k$ for $k \neq i, j$ we see that z_k are positive numbers, $z_1 + \ldots + z_n = x_1 + \ldots + x_n = 1$, and $f(z_1, \ldots, z_n) \leq f(x_1, \ldots, x_n) = M$. Since M is the smallest value of f, we must have $f(z_1, \ldots, z_n) = M$, which is equivalent to $\left\lfloor \frac{1}{x_j + d} \right\rfloor = \left\lfloor \frac{1}{x_j} \right\rfloor - 1$. This means that $(z_1, \ldots, z_n) \in \mathcal{F}$, as claimed in 3.

Yuqiao's key idea is to start with a point in $\mathcal{F}$ and keep modifying it using 3. until we get a point $(y_1, ..., y_n)$ such that $\left\lfloor \frac{1}{y_i} \right\rfloor \in \{n-1, n\}$ for every i. To make this precise, for any $(x_1, \ldots, x_n) \in \mathcal{F}$ set $l(x_1, \ldots, x_n)$ to be the sum of all those numbers $\left\lfloor \frac{1}{x_i} \right\rfloor$ which are bigger than n (we set $l(x_1, \ldots, x_n) = 0$ if no such number exists). Pick $(u_1, \ldots, u_n) \in \mathcal{F}$ for which $l = l(u_1, \ldots, u_n)$ is smallest possible. If $l \neq 0$ then $\left\lfloor \frac{1}{u_j} \right\rfloor > n$ for some j. By property 2. we have $\left\lfloor \frac{1}{u_i} \right\rfloor \leq n-1$ for some i. Setting $z_i = u_i - d(u_i, u_j)$, $z_j = u_j + d(u_i, u_j)$, $z_k = u_k$ for $k \neq i, j$ we get from 3. that $(z_1, \ldots, z_n) \in \mathcal{F}$, $\left\lfloor \frac{1}{z_j} \right\rfloor = \left\lfloor \frac{1}{u_j} \right\rfloor - 1$, $\left\lfloor \frac{1}{z_i} \right\rfloor = \left\lfloor \frac{1}{u_i} \right\rfloor + 1$, and $\left\lfloor \frac{1}{z_k} \right\rfloor = \left\lfloor \frac{1}{u_k} \right\rfloor$ for $k \neq i, j$. It is clear now that $l(z_1, \ldots, z_n) < l(u_1, \ldots, u_n)$ contradicting our choice of $(u_1, ..., u_n)$. This shows that we must have $l = 0$, i.e. $\left\lfloor \frac{1}{u_k} \right\rfloor \leq n$ for every k.

Now among all $(u_1, \ldots, u_n) \in \mathcal{F}$ such that $\left\lfloor \frac{1}{u_k} \right\rfloor \leq n$ for every k (we just proved such points exist) choose one $(y_1, \ldots, y_n)$ with the smallest number of coordinates y_k such that $\left\lfloor \frac{1}{y_k} \right\rfloor = n$. There is at least one such coordinate, say y_j, by property 1. We claim that $\left\lfloor \frac{1}{y_k} \right\rfloor \geq n-1$ for all k. Indeed, suppose that $\left\lfloor \frac{1}{y_i} \right\rfloor < n-1$ for some i. Setting $z_i = y_i - d(y_i, y_j)$, $z_j = y_j + d(y_i, y_j)$, $z_k = y_k$ for $k \neq i, j$ we get from 3. that $(z_1, \ldots, z_n) \in \mathcal{F}$, $\left\lfloor \frac{1}{z_j} \right\rfloor = \left\lfloor \frac{1}{y_j} \right\rfloor - 1 = n - 1$, $\left\lfloor \frac{1}{z_i} \right\rfloor = \left\lfloor \frac{1}{y_i} \right\rfloor + 1 \leq n-1$, and $\left\lfloor \frac{1}{z_k} \right\rfloor = \left\lfloor \frac{1}{y_k} \right\rfloor$ for $k \neq i, j$. It is clear now that $(z_1, \ldots, z_n)$ has fewer coordinates whose reciprocal has floor equal to n than $(y_1, \ldots, y_n)$, contradicting our choice of $(y_1, ..., y_n)$. The contradiction proves that $\left\lfloor \frac{1}{y_k} \right\rfloor \in \{n-1, n\}$ for every k. This completes our argument.