Problem 3. Jack and Jill play the following game which results in a 6 digit number: Jill starts by picking a non-zero digit, the first digit of the number. Then Jack and Jill alternate picking the next digits, each time they can choose any digit which has not been used before. Jack wins if the 6 digit number is a prime, Jill wins otherwise. For example, suppose Jill picks 8, then Jack picks 0, then Jill picks 9, then Jack picks 4, then Jill picks 6, and finally Jack picks 1. We get the number 809461, which is a prime number, so Jack wins. Which player has a strategy to win regardless of how the other plays?

Solution. Let us start by recalling some elementary facts about divisibility. If the last digit of a number is even, then the number itself is even. If the last digit is 5 then the number is divisible by 5. If the sum of the digits is divisible by 3 then the number is also divisible by 3.

We will show that Jill has a strategy to win. Divide our digits into 2 sets: $A = \{1, 3, 7, 9\}$, $B = \{0, 2, 4, 5, 6, 8\}$. Note that Jack in his last move has to choose a number from A, otherwise he will loose regardless of any other moves (as the 6 digit number will be either even or divisible by 5).

Jill starts by choosing 3. If Jack chooses on his first move a number from A, Jill will keep choosing numbers from A if possible on her next moves and on Jack's last move no number from A will be available, so Jack will loose. Thus Jack must choose a digit m from B. Then Jill chooses 9 on her second move. Again, if Jack chooses from A on his second move, Jill will choose the remaining 4th digit from A on her third move and Jack will have no number from A left on his last move, so he will loose. Thus Jack must choose a digit n from B on his second move. On his last move Jack will have to choose either 1 or 7. Each of these 2 possibilities leaves reminder 1 modulo 3. Thus Jill on her third move should choose a digit d such that $3 + m + 9 + n + d + 1$ is divisible by 3, i.e. such that $m + n + 1 + d$ is divisible by 3 (then the six digit number will be divisible by 3 and Jack will loose). Jill can indeed do that. In fact, if $m + n + 1$ is divisible by 3 then at least one of m, n is not divisible by 3, so one of $0, 6$ is still available and Jill chooses it on her last move. If $m + n + 1$ leaves reminder 1 modulo 3, then Jill should choose any remaining number from $\{2, 5, 8\}$ (at least one is still available). Finally, if $m + n + 1$ leaves reminder 2 modulo 3, Jill should choose 1.