Problem 3. An outcome of flipping a coin n times is called k-lucky if it contains a pattern which is repeated k times in a row. For example, the outcome THHTHTHTTH (T stands for "tales" and H for "heads") of flipping a coin 10 times is 3-lucky since HT appears 3 times in a row. Let P_n be the probability that flipping a coin n times is 6-lucky. Find t as small as you can so that $P_n < t$ for all n.

Solution. Let L_n be the set of all 6-lucky outcomes when a coin is flipped n times and let U_n be the set of all outcomes which are not 6-lucky. Then $|L_n|+|U_n| = 2^n$ and $P_n = |L_n|/2^n = 1-|U_n|/2^n$. Note that $|U_k| = 2^k$ for $k \leq 5$ but $L_6 = 2$ (there are exactly two 6-lucky outcomes of 6 throws: $HHHHHH$ and $TTTTTT$) so $|U_6| = 62$.

Now conisder an element in U_{2n}. The first n throws and the last n throws provide outcomes in U_n. This menas that $|U_{2n}| \leq |U_n| \times |U_n|$. By a straigformard induction $|U_{6\times 2^k}| \leq |U_6|^{2^k} = 60^{2^k}$ for all k. Thus

$$P_{6\times 2^k} \geq 1 - \frac{60^{2^k}}{2^{6\times 2^k}} = 1 - \left(\frac{31}{32}\right)^{2^k}.$$

Since the quantity $(31/32)^{2^k}$ approaches 0 as k tends to infinity, we see that no $t < 1$ will work. Thus the best t which works is $t = 1$.

This is essentially the solution submitted by Yuqiao Huang. One can improve it slightly by observing that for any element of U_{m+k}, the first m throws form an elelemnt in U_m and the last k throws form an element in U_k, so $|U_{m+k}| \leq |U_m| \times |U_k|$. It follows easily that $|U_{6k+i}| \leq |U_6|^k \times |U_i| = 62^k \cdot 2^i$, for $i = 0, 1, 2, 3, 4, 5$. Thus, for $n = 6k + i$ we have

$$P_n \geq 1 - \frac{62^k \cdot 2^i}{2^{6k+i}} = 1 - \left(\frac{31}{32}\right)^k \geq 1 - \left(\frac{31}{32}\right)^{n/6}$$

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A little more challenging is the following problem.

Problem. Show that there is $c > 0$ such that $P_n < 1 - c^n$ for all n (or for all sufficiently large n) and find such a c as large as you can.