Today's plan:

- Section 2.3: Equal values - Different rights
- Section 2.3.1: Hamilton's method.

Section 2.3: Equal values - Different rights

Example

- Joe and Mary buy a lottery ticket

Example

- Joe and Mary buy a lottery ticket
- Mary contributed $10 and Joe contributed $5

Example

- Joe and Mary buy a lottery ticket
- Mary contributed $10 and Joe contributed $5
- They win $500.00

Solution

- *Mary deserves twice as much as Joe*

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- *So she gets \$333.33 and he gets \$166.67*

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- *Mary deserves twice as much as Joe*
- *So she gets $333.33 and he gets $166.67*

Just like with equal rights, continuous problems are straightforward. Just divide the item into parts of the right size and you're done.

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- This is the **apportionment** problem.

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- The Senate, consisting of 2 senators per state

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- The Senate, consisting of 2 senators per state
- The House of Representatives, whose seats *"shall be apportioned among the several States according to their respective numbers"*

The Convention left it to Congress to decide how to do this.

Example

In 1790, Alexander Hamilton proposed that everyone use Hamilton's method.

We'll study Hamilton's method in Section 2.3.1.

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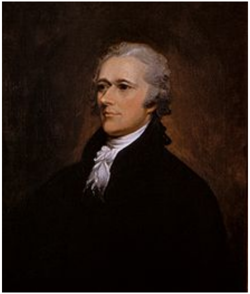

Alexander Hamilton (1755 - 1804)

Let's consider a small example first

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Example

- Mr. Jones gives a bag of 19 lollipops to his three kids.

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Example

- Mr. Jones gives a bag of 19 lollipops to his three kids.
- Since Mary is older than twins Joe and Ann, Mary deserves twice as many lollipops.

Solution

- *Mary should get half of the lollipops* (**9.5**)

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- *Mary should get half of the lollipops* (**9.5**)
- *The twins should get one quarter* (**4.75**) *each*

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- *Mary should get half of the lollipops* $(\mathbf{9.5})$
- *The twins should get one quarter* $(\mathbf{4.75})$ *each*

So of the 19 lollipops, Mary gets 9, Joe gets 4, Ann gets 4....

Solution

- *....leaving them with a surplus of 2 lollipops.*

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- *Both Joe and Ann were closer to 5 than Mary was to 10, so they decide Joe and Ann each get 1 surplus lollipop.*

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- *....leaving them with a surplus of 2 lollipops.*
- *Both Joe and Ann were closer to 5 than Mary was to 10, so they decide Joe and Ann each get 1 surplus lollipop.*

They end up with:

Mary: 9 *Joe: 5* *Ann: 5*

- They actually applied **Hamilton's method** for apportionment.

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- The idea is that you assign the surplus lollipops to the two kids with the "highest fractional part" in their "quota."

Let's shift to the relevant terminology:

- The items to be apportioned are called **seats**

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- The number of seats is denoted by M.
- The recipients of seats are called **states**.
- The numbers that determine the apportionment are called the **population**.

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$$\text{lollipops} \leftrightarrow \text{House seats}$$

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lollipops $\leftrightarrow$ House seats

$M = 19$

Mary, Joe and Ann $\leftrightarrow$ states

Ages $\leftrightarrow$ populations

Section 2.3.1: Hamilton's method.

Example

- The Clearview Chamber of Commerce donates 100 new computers to the school district.

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- The Clearview Chamber of Commerce donates 100 new computers to the school district.
- The computers should be apportioned based on the schools' enrollment.

Enrollment:

School	A	B	C	D	E	F	**Total**
Enrollment	251	379	154	228	195	217	1424

- Since the total enrollment is 1424 students, one computer should be allocated for every 14.24 students.

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- We call this number the **standard divisor** and we denote it by the letter **d**:

$$d = \frac{\text{total enrollment}}{\text{number of computers}} = \frac{1424}{100} = 14.24$$

With the standard divisor $d = 14.24$, we get the **exact quota** for each school.

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Exact quota: $q = \dfrac{\text{enrollment}}{\text{standard divisor}}$

Note: Since $d = \dfrac{\text{total enrollment}}{\text{total \# computers}}$

and $q = \dfrac{\text{individual enrollment}}{d}$,

we also have:

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and $q = \dfrac{\text{individual enrollment}}{d}$,

we also have:

$q = (\%\text{ of students}) \cdot (\#\text{ computers})$.

School	A	B	C	D	E	F	Total
Enrollment	251	379	154	228	195	217	1424
No. computers:	100			Std. divisor:	14.24		
Exact Q.	17.626	26.615	10.815	16.011	13.694	15.239	100

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Definition

- The **integral part** (before the decimal point) of the exact quota is the **lower quota**.

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- The **integral part** (before the decimal point) of the exact quota is the **lower quota**.
- The **fractional part** is what's left after the decimal point.

- We first allocate to each school its lower quota.

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- That adds up to 97 computers, leaving a surplus of 3 computers.

Question

What do we do with the remaining 3 computers?

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Answer: Assign them to the three schools with the highest fractional part.

School	A	B	C	D	E	F	**Total**
Enrollment	251	379	154	228	195	217	1424
No. computers:		100		**Std. divisor:**		14.24	
Exact Q.	17.626	26.615	10.815	16.011	13.694	15.239	100
Lower Q.	17	26	10	16	13	15	97
Fract.	.626	.615	.815	.011	.694	.239	
Surplus							3
Total							

School	A	B	C	D	E	F	**Total**
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Surplus			1 (1^{st})				3
Total							

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Lower Q.	17	26	10	16	13	15	97
Fract.	.626	.615	.815	.011	.694	.239	
Surplus			1 (1^{st})		1 (2^{nd})		3
Total							

School	A	B	C	D	E	F	**Total**
Enrollment	251	379	154	228	195	217	1424
No. computers:		100			**Std. divisor:**	14.24	
Exact Q.	17.626	26.615	10.815	16.011	13.694	15.239	100
Lower Q.	17	26	10	16	13	15	97
Fract.	.626	.615	.815	.011	.694	.239	
Surplus	1 (3^{rd})		1 (1^{st})		1 (2^{nd})		3
Total							

School	A	B	C	D	E	F	**Total**
Enrollment	251	379	154	228	195	217	1424
No. computers:		100		**Std. divisor:**		14.24	
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Lower Q.	17	26	10	16	13	15	97
Fract.	.626	.615	.815	.011	.694	.239	
Surplus	1 (3^{rd})		1 (1^{st})		1 (2^{nd})		3
Total	18	26	11	16	14	15	100

- Now we formulate the method precisely.

- Now we formulate the method precisely.
- In legislature terminology:

$$\text{computers} \leftrightarrow \text{House seats}$$
$$\text{schools} \leftrightarrow \text{states}$$
$$\text{enrollment} \leftrightarrow \text{population}$$

Terms that we need:

Standard Divisor:

$$d = \frac{\text{total population}}{\text{number of seats}}$$

State's Exact Quota:

$$q = \frac{\text{State's Population}}{\text{Standard Divisor}}$$

Surplus Seats:

$$\text{Number of seats} - \sum(\text{lower quotas})$$

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$$\text{Number of seats} - \sum(\text{lower quotas})$$

Note that a **surplus** means that not all seats have been allocated.

Hamilton's Method

1. Compute the standard divisor, and determine each state's exact quota.
2. Split each exact quota into lower quota and fractional part.
3. Find the total surplus. Allocate the surplus seats according to the highest fractional part.
4. Each state's final allocation is the lower quota, possibly plus one surplus seat from step 3.

Back to the original example of the House of Representatives.

Example (cont'd)

- In 1790 the U.S. population was $3,615,920$

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- and there were 15 states.

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- Alexander Hamilton proposed a House with 120 seats, apportioned using his method.

Example (cont'd)

- In 1790 the U.S. population was $3,615,920$
- and there were 15 states.
- Alexander Hamilton proposed a House with 120 seats, apportioned using his method.

Find this apportionment.

With a total population of 3,615,920 people, and a House of 120 seats, the standard divisor is:

$$d = \frac{3,615,920}{120} = 30,132.667$$

State	Population	Exact quota	Lower quota	Frac. part	Surplus		Total
M=120	**d**=30,132.667						
Virginia	630,560	20.926	20	.926	1	2nd	21
Massachusetts	475,327	15.774	15	.774	1	7th	16
Pennsylvania	432,879	14.366	14	.366			14
North Carolina	353,523	11.732	11	.732	1	8th	12
New York	331,589	11.004	11	.004			11
Maryland	278,514	9.243	9	.243			9
Connecticut	236,841	7.860	7	.860	1	3rd	8
South Carolina	206,236	6.844	6	.844	1	4th	7
New Jersey	179,570	5.959	5	.959	1	1st	6
New Hampshire	141,822	4.707	4	.707	1	9th	5
Vermont	85,533	2.839	2	.839	1	6th	3
Georgia	70,835	2.351	2	.351			2
Kentucky	68,705	2.280	2	.280			2
Rhode Island	68,446	2.271	2	.271			2
Delaware	55,540	1.843	1	.843	1	5th	2
Total	3,615,920	119.999	111		9		120

Next time:
Section 2.3.2: Jefferson's method.