Chapter II

Fair Division - Apportionment

Today's plan:

- Section 2.1: Defining the problem
- Section 2.2: Equal values – Equal rights

Section 2.1: Defining the problem

Historical Examples

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King Solomon:

Two women each claim the same baby. Rather than apportion one baby to one woman and zero babies to the other, Solomon says he'll apportion half a baby to each.

One woman agrees, but the other doesn't, and offers it to the first, as she would rather the baby live. So clearly the second woman is the mother.

Aesop:

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A lion, a fox, and a donkey go hunting.

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A lion, a fox, and a donkey go hunting. The donkey divides the kill into three equal shares.

Enraged at only being given one third, the lion eats the donkey, and tells the fox to divide it up differently.

The fox divides it into one huge share and one tiny share.

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The lion is the only one who is happy, though. The fox got only a tiny piece, and the donkey is dead.

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Note that the first two can't be divided up, but the last one can.

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Some of the results even turned out to be worth a Nobel prize.

Defining the Problem

Example (motivating)

- Joe and Mary buy a lottery ticket together and win $500.00. They want to split the money.

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- The U.S. House of Representatives consists of 435 members. According to the Constitution, the seats *"shall be apportioned among the several States according to their respective numbers"*.

Example (Continued)

- Ann and Bob inherit a farm from their parents. They want to divide it into two parcels.

Example (Continued)

- The four owners of The Kleen Car Wash Co. unanimously decide to dissolve the company.

Example (Continued)

- The four owners of The Kleen Car Wash Co. unanimously decide to dissolve the company. Each of them will get the appropriate percentage of assets, based on the percentage of company stock they own.

Classifying types of problems

First, fair division problems can be classified by the **type of items to be divided**.

In the first and third examples (money and farm), the items can be split arbitrarily.

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- Joe and Mary can split the money into two shares any way (up to a penny).
- The boundary between the two plots of land can be drawn anywhere.

Definition

We call these **continuous** fair division problems. (Even though we can't split a penny.)

In the second example (congressional seats), the items to be divided cannot be cut into pieces.

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- A state gets apportioned a full house seat, not a fraction of it.

Definition

We call this a **discrete** fair division problem.

In the fourth example,

- The Kleen Car Wash has some assets that cannot be split, like cars and sponges.

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- It also has assets that can be divided arbitrarily, like money.

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- The Kleen Car Wash has some assets that cannot be split, like cars and sponges.
- It also has assets that can be divided arbitrarily, like money.
- Thus we have some discrete elements and some continuous elements.

Definition

We call this a **mixed** fair division problem.

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- Whenever dealing with cash, all parties agree on its value.
- A dollar is worth a dollar.

Definition

Fair division problems where all the parties agree on the value of each item are called **equal values** problems.

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- Bob may value the land near the river more

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- Bob may value the land near the river more
- Ann may value the land by the road more.

Definition

When the parties don't agree on the value of the items, we call such problems **different values** problems.

Example

- In the House of Representatives the House seats are indistinguishable.

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Example

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- In The Kleen Car Wash Company, the parties may value the assets differently. This is a different values problem.

Classifying types of problems

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Finally, fair division problems can be classified according to the **parties' rights**.

- All parties may be entitled to equal shares of the division, like the first and third examples (money/land).
- Or they may be entitled to different size shares, like in the second and fourth examples.

Definition

- We call the first **equal rights** fair division problems.

Definition

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- We call the second **different rights** fair division problems.

Question

What does fair division mean?

Definition

A division is a **fair division** if everyone ends up with a share that he/she considers to be worth at least what he/she was entitled to.

Remarks:

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- In a fair division, it is possible that someone, **A**, is mad because she thinks someone else, **C**, got a better share.
- That is not a concern for fair division; as long as **A** considers her piece to be worth at least what she is entitled to, it's fair.

Definition

If in addition to everyone getting what they consider fair, they also feel that nobody else fared better then they deserved, we have what's called an **envy-free division**.

- Getting envy-free divisions is a lot harder than getting fair divisions.

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- We are mainly concerned with fair division methods, not with envy-free divisions.

Section 2.2: Equal Values - Equal Rights

Example

Joe and Mary buy a lottery ticket together (equal parts) and win $500.00. They are going to split the money.

Solution

- *Nothing could be easier*

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Solution

- *Nothing could be easier*
- *They value money equally*
- *They are entitled to equal parts*
- *Money's divisible, so they split it into two shares of $250.00 each.*

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- Just divide the item into equal parts and you're done.

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Example

Mr. Jones gives a bag of lollipops to his three kids.

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Example

Mr. Jones gives a bag of lollipops to his three kids. He tells them to divide the lollipops equally among the three of them.

Solution

- *It turns out that the bag had 19 lollipops - not divisible by 3.*

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- *Sharing lollipops is gross.*

Solution

Thus we have a discrete fair division problem.

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The kids make the wise decision to take 6 lollipops each, and give the remaining 1 to their father.

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Thus we have a discrete fair division problem.
The kids make the wise decision to take 6 lollipops each, and give the remaining 1 to their father. How sweet.

Remarks:

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- Each kid feels she/he is entitled to 6 and 1/3 lollipops, but since lollipops are unbreakable, getting 6 is the closest they can get.
- Everyone's happy.

Instead of giving the last lollipop to their father, they could have drawn straws.

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One would have ended up with 7, and the other two with 6.

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One would have ended up with 7, and the other two with 6.
That would not be an envy-free division, but still is fair.

Equal-values-equal-rights, discrete fair division problems call for

- dividing the items into equal numbers
- randomly allocating any remaining items

Next time:
Section 2.3: Equal values – Different rights; Apportionment
and
Section 2.3.1: Hamilton's method.