Section 1.3: One person - Multiple votes; Two alternatives
...Continued!
Example

- The United Nations Security Council is composed of 15 members.
Example

- The United Nations Security Council is composed of 15 members.
- 5 permanent:
Example

- The United Nations Security Council is composed of 15 members.
- 5 permanent: China, France, Russia, UK and US.
Example

- The United Nations Security Council is composed of 15 members.
- 5 permanent: China, France, Russia, UK and US.
- And 10 elected to two-year terms.
Example

The current 10 non-permanent members are: Angola, Chad, Chile, Jordan, Lithuania, Malaysia, New Zealand, Nigeria, Spain, Venezuela.
Example

Votes on

- **procedural matters** require “yes” votes from **9 members**
Example

Votes on

- **procedural matters** require “yes” votes from 9 members
- **substantive matters** require “yes” votes from all five permanent, plus four non-permanent members.

Describe these as weighted voting systems.
Example

Votes on

- **procedural matters** require “yes” votes from 9 members
- **substantive matters** require “yes” votes from all five permanent, plus four non-permanent members.

Describe these as weighted voting systems.
For procedural matters, all 15 countries are equal and they need 9 “yes” votes:
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[9 : 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
For substantive matters it’s more complicated.

Want (10 non-pmt) + (4 pmt) < q,
but also (4 non-pmt) + (5 pmt) = q.
I claim:

\[39 : 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\]

works.
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\[39 : 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\]

works.

Bonus problem: Solve this with algebra.
I claim:

\[39 : 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\]

works.

Bonus problem: Solve this with algebra. (Disclaimer: Not actually for bonus points.)
Let’s check:

\[39 : 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\]

- All 10 non-permanent members plus 4 permanent members gives 38 votes; not enough.
Let’s check:

\[39 : 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\]

- All 10 non-permanent members plus 4 permanent members gives 38 votes; not enough.
- Each permanent member has veto power.
[39 : 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]

On the other hand, the 5 permanent members plus 4 non-permanent members gives 39 votes; enough.
Next:

- Section 1.3.1: Coalitions
- Section 1.3.2: Critical voters; Power Index
Section 1.3.1 : Coalitions
In the previous example, instead of telling us the weights and the quotas, the problem told us which groups of voters will/won’t win.
In the previous example, instead of telling us the weights and the quotas, the problem told us which groups of voters will/won’t win.

In other words it described the winning coalitions.
A coalition is a group of voters that may join together to vote.

A winning coalition is called a winning coalition if it has enough votes to meet the quota.
Definition

A coalition is a group of voters that may join together to vote.

A coalition is called a winning coalition if it has enough votes to meet the quota.
Example

Determine the winning coalitions in the Kleen Car Wash Co. voting system for special propositions:

\[7 : 4, 3, 2, 1\]
Our Strategy:

1. List all of the coalitions.
2. Find the total weight of each coalition.
3. Decide which of the coalitions are winners.
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<table>
<thead>
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Remarks:

- There are a total of 15 coalitions

- Only 5 of which are winning

- Notice that A is a member of every winning coalition. This means that A has veto power!

- On the other hand, having A in a coalition is not enough to make it a winning coalition. This means that A is not a dictator.
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Section 1.3.2 : Critical Voter; Power Index
Remark

Despite the fact that $C$ has twice as many votes as $D$, they both appear the same number of times (3) in winning coalitions.
We might be tempted to say that they have the same amount of power...
We might be tempted to say that they have the same amount of power...

This not quite the right way to look at power.
We might be tempted to say that they have the same amount of power...

This not quite the right way to look at power.

Just being part of a winning coalition doesn’t give you power
It is the ability to threaten to break up a winning coalition.
A member may say to his coalition allies

“I’ll vote yes on issue X, but only if you support me on issue Y.”
The rest of the coalition members can decide if they really need him to win.
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If they don’t need him, then his threat is pointless.
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If they don’t need him, then his threat is pointless.

But if the withdrawal of that voter from the coalition turns it into a losing one, then he/she has power.
We call such a voter a critical voter in that coalition.
More precisely,

**Definition**

If a voter $A$ is a member of a winning coalition, but the coalition obtained by removing $A$ is not a winning coalition, then $A$ is called a critical voter for that coalition.
Back to the Kleen Car Wash Co.

[7 : 4, 3, 2, 1]

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Example

- In the winning coalition \(\{A, B, D\}\), both A and B are critical voters.
Example

In the winning coalition \{A, B, D\} both A and B are critical voters, since both \{B, D\} and \{A, D\} are losing coalitions.
Example

- In the winning coalition \( \{A,B,D\} \) both A and B are critical voters, since both \( \{B,D\} \) and \( \{A,D\} \) are losing coalitions.
- D is not critical:
Example

- In the winning coalition \( \{A, B, D\} \) both A and B are critical voters, since both \( \{B, D\} \) and \( \{A, D\} \) are losing coalitions.
- D is not critical: if D withdraws, the remaining coalition \( \{A, B\} \) still wins.
Example

Find all the critical voters in the winning coalitions of the Kleen Car Wash Co. voting system for special propositions.
Let’s list the winning coalitions:

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And mark the critical voters:

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</table>
A has 5 occurrences as a critical voter;
A has 5 occurrences as a critical voter; B has 3 occurrences;
A has 5 occurrences as a critical voter; B has 3 occurrences; C has 1
A has 5 occurrences as a critical voter; B has 3 occurrences; C has 1 and D has 1.
For each voter record the number of occurrences as a critical voter as a percentage of the total number of critical voter occurrences. This is called the **power index**.
That is:

Power index of $X =$

\[
\frac{\text{number of occurrences of } X \text{ as a critical voter}}{\text{total number of critical voter occurrences}}
\]
Example

Find the power index for the Kleen Car Wash Co. voting system for special propositions.
Solution

There are a total of 10 occurrences of critical voters so the power index is:

\[ A : \frac{5}{10} = 50\% \]

\[ B : \frac{3}{10} = 30\% \]

\[ C : \frac{1}{10} = 10\% \]

\[ D : \frac{1}{10} = 10\% \]
Observe:

- It makes sense A has more power and D has less than the others.
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- It’s somewhat unexpected that C and D have the same power, since C has twice as many votes as D.
Observe:

- It makes sense \textbf{A} has more power and \textbf{D} has less than the others.
- It’s somewhat \textit{unexpected} that \textbf{C} and \textbf{D} have the same power, since \textbf{C} has twice as many votes as \textbf{D}.
- There is a correlation between weight and power, but not a perfect match.
There are other power indices to measure the power of voters. This one is called the Penrose-Banzhaf power index or just the Banzhaf power index.
It was created by the mathematician Lionel Penrose in 1946.
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John Banzhaf (1940 - )
Next time:
Section 1.4. : One Person – Multiple Votes; Multiple Alternatives and Section 1.5. : Breaking Ties.