

# Section 1.3 : One person - Multiple votes; Two alternatives ...Continued!

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- ▶ 5 permanent: China, France, Russia, UK and US.
- ▶ And 10 elected to two-year terms.

### Example

The current 10 non-permanent members are: Angola, Chad, Chile, Jordan, Lithuania, Malaysia, New Zealand, Nigeria, Spain, Venezuela.

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Votes on

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Votes on

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- ▶ **substantive matters** require “yes” votes from **all five permanent, plus four non-permanent members.**

Describe these as weighted voting systems.

For procedural matters, all 15 countries are equal and they need 9 “yes” votes:

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[9 : 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]

For substantive matters it's more complicated.

Want  $(10 \text{ non-pmt}) + (4 \text{ pmt}) < q$ ,

but also  $(4 \text{ non-pmt}) + (5 \text{ pmt}) = q$ .

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[39 : 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1]

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Bonus problem: Solve this with algebra. (Disclaimer: Not actually for bonus points.)

Let's check:

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- ▶ All 10 non-permanent members plus 4 permanent members gives 38 votes; not enough.



Let's check:

[39 : 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]

- ▶ All 10 non-permanent members plus 4 permanent members gives 38 votes; not enough.
- ▶ Each permanent member has veto power.

[39 : 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1]

On the other hand, the 5 permanent members plus 4 non-permanent members gives 39 votes; enough.

# Next:

- ▶ Section 1.3.1 : Coalitions
- ▶ Section 1.3.2 : Critical voters;  
Power Index

## Section 1.3.1 : Coalitions

In the previous example, instead of telling us the weights and the quotas, the problem told us which groups of voters will/won't win.

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In other words it described the **winning coalitions**.

### Definition

A **coalition** is a group of voters that may join together to vote.

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A **coalition** is a group of voters that may join together to vote.

A coalition is called a **winning coalition** if it has enough votes to meet the quota.



### Example

Determine the winning coalitions in the Kleen Car Wash Co. voting system for special propositions:

$$[7 : 4, 3, 2, 1]$$

## Our Strategy:

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2. Find the total weight of each coalition.
3. Decide which of the coalitions are winners.

Coalition	weight	W/L	Coalition	weight	W/L
{A}			{B,D}		
{B}			{C,D}		
{C}			{A,B,C}		
{D}			{A,B,D}		
{A,B}			{A,C,D}		
{A,C}			{B,C,D}		
{A,D}			{A,B,C,D}		
{B,C}					

Coalition	weight	W/L	Coalition	weight	W/L
{A}	4		{B,D}	4	
{B}	3		{C,D}	3	
{C}	2		{A,B,C}	9	
{D}	1		{A,B,D}	8	
{A,B}	7		{A,C,D}	7	
{A,C}	6		{B,C,D}	6	
{A,D}	5		{A,B,C,D}	10	
{B,C}	5				

Coalition	weight	W/L	Coalition	weight	W/L
{A}	4	L	{B,D}	4	L
{B}	3	L	{C,D}	3	L
{C}	2	L	{A,B,C}	9	<b>W</b>
{D}	1	L	{A,B,D}	8	<b>W</b>
{A,B}	7	<b>W</b>	{A,C,D}	7	<b>W</b>
{A,C}	6	L	{B,C,D}	6	L
{A,D}	5	L	{A,B,C,D}	10	<b>W</b>
{B,C}	5	L			



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## Remarks:

- ▶ There are a total of 15 coalitions
- ▶ Only 5 of which are winning
- ▶ Notice that **A** is a member of every winning coalition. This means that **A** has veto power!
- ▶ On the other hand, having **A** in a coalition is not enough to make it a winning coalition. This means that **A** is not a dictator.

## Section 1.3.2 : Critical Voter; Power Index

## Remark

*Despite the fact that **C** has twice as many votes as **D**, they both appear the **same number of times (3)** in winning coalitions.*



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- ▶ This not quite the right way to look at power.
- ▶ Just being part of a winning coalition doesn't give you power

It is the ability to **threaten** to break up a winning coalition.

A member may say to his coalition allies

*“I’ll vote yes on issue X, but only if you support me on issue Y.”*

- ▶ The rest of the coalition members can decide if they really need him to win.

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- ▶ If they don't need him, then his threat is pointless.

- ▶ The rest of the coalition members can decide if they really need him to win.
- ▶ If they don't need him, then his threat is pointless.
- ▶ But if the withdrawal of that voter from the coalition turns it into a losing one, then he/she has **power**.



We call such a voter a **critical voter** in that coalition.

More precisely,

#### Definition

If a voter **A** is a member of a winning coalition, but the coalition obtained by removing **A** is not a winning coalition, then **A** is called a critical voter for that coalition.

Back to the Kleen Car Wash Co.

[7 : 4, 3, 2, 1]

Coalition	weight	W/L	Coalition	weight	W/L
{A}	4	L	{B,D}	4	L
{B}	3	L	{C,D}	3	L
{C}	2	L	{A,B,C}	9	<b>W</b>
{D}	1	L	{A,B,D}	8	<b>W</b>
{A,B}	7	<b>W</b>	{A,C,D}	7	<b>W</b>
{A,C}	6	L	{B,C,D}	6	L
{A,D}	5	L	{A,B,C,D}	10	<b>W</b>
{B,C}	5	L			

### Example

- ▶ In the winning coalition  $\{\mathbf{A}, \mathbf{B}, \mathbf{D}\}$  both  $\mathbf{A}$  and  $\mathbf{B}$  are critical voters,

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- ▶  $\mathbf{D}$  is not critical: if  $\mathbf{D}$  withdraws, the remaining coalition  $\{\mathbf{A}, \mathbf{B}\}$  still wins.

### Example

Find all the critical voters in the winning coalitions of the Kleen Car Wash Co. voting system for special propositions.



## Solution

*Let's list the winning coalitions:*

<i>Coalition</i>	<i>weight</i>	<i>W/L</i>
$\{A, B\}$	7	<b>W</b>
$\{A, B, C\}$	9	<b>W</b>
$\{A, B, D\}$	8	<b>W</b>
$\{A, C, D\}$	7	<b>W</b>
$\{A, B, C, D\}$	10	<b>W</b>

## Solution

*And mark the critical voters:*

<i>Coalition</i>	<i>weight</i>	<i>W/L</i>
$\{A, B\}$	7	<b>W</b>
$\{A, B, C\}$	9	<b>W</b>
$\{A, B, D\}$	8	<b>W</b>
$\{A, C, D\}$	7	<b>W</b>
$\{A, B, C, D\}$	10	<b>W</b>

**A** has 5 occurrences as a critical voter;

**A** has 5 occurrences as a critical voter; **B** has 3 occurrences;

**A** has 5 occurrences as a critical voter; **B** has 3 occurrences;  
**C** has 1

**A** has 5 occurrences as a critical voter; **B** has 3 occurrences; **C** has 1 and **D** has 1.

### Definition

For each voter record the number of occurrences as a critical voter as a percentage of the total number of critical voter occurrences. This is called the **power index**.

That is:

Power index of  $X =$

$$\frac{\text{number of occurrences of } X \text{ as a critical voter}}{\text{total number of critical voter occurrences}}$$



### Example

Find the power index for the Kleen Car Wash Co. voting system for special propositions.

### Solution

*There are a total of 10 occurrences of critical voters so the power index is:*

$$A : \frac{5}{10} = 50\% \quad B : \frac{3}{10} = 30\%$$
$$C : \frac{1}{10} = 10\% \quad D : \frac{1}{10} = 10\%$$

## Observe:

- ▶ It makes sense **A** has more power and **D** has less than the others.

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## Observe:

- ▶ It makes sense **A** has more power and **D** has less than the others.
- ▶ It's somewhat **unexpected** that **C** and **D** have the same power, since **C** has twice as many votes as **D**.
- ▶ There is a correlation between weight and power, but not a perfect match.

There are other power indices to measure the power of voters. This one is called the **Penrose-Banzhaf power index** or just the **Banzhaf power index**.

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John Banzhaf (1940 - )

Next time:  
Section 1.4. : One Person – Multiple Votes;  
Multiple Alternatives and Section 1.5. :  
Breaking Ties.