

Today's plan:

Section 1.2.5 : Pairwise Comparisons Method, Condorcet criterion, Independence of irrelevant alternatives criterion, and Section 1.2.6 : Summary of voting methods and fairness criteria.

In the **Pairwise Comparisons method**:

- ▶ Compare each pair of candidates

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In the **Pairwise Comparisons method**:

- ▶ Compare each pair of candidates
- ▶ The winner of each comparison gets 1 point
- ▶ In a tie, each of the two candidates gets $1/2$ point.
- ▶ Whoever has the most points overall wins

Example

Find the winner of the Math Club presidential election using the Pairwise Comparisons method.

Choice	Number of ballots				
	8	6	1	1	4
1st	A	C	C	B	B
2nd	B	D	D	D	C
3rd	C	B	A	C	D
4th	D	A	B	A	A

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A							
B							
C							
D							

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A							
B							
C							
D							

A v/s B.

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A							
B							
C							
D							

A v/s B.

- ▶ In columns 1 and 3: $A > B$ (9 votes)

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A							
B							
C							
D							

A v/s B.

- ▶ In columns 1 and 3: $A > B$ (9 votes)
- ▶ In columns 2,4 and 5: $B > A$ (11 votes)

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A							
B							
C							
D							

A v/s B.

- ▶ In columns 1 and 3: $A > B$ (9 votes)
- ▶ In columns 2,4 and 5: $B > A$ (11 votes)
- ▶ So A gets 0 points, B gets 1.

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0						
B	1						
C							
D							

A v/s B.

- ▶ In columns 1 and 3: $A > B$ (9 votes)
- ▶ In columns 2,4 and 5: $B > A$ (11 votes)
- ▶ So A gets 0 points, B gets 1.

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0						
B	1						
C							
D							

A v/s C.

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0						
B	1						
C							
D							

A v/s C.

- ▶ In column 1: $A > C$ (8 votes)

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0						
B	1						
C							
D							

A v/s C.

- ▶ In column 1: $A > C$ (8 votes)
- ▶ In columns 2,3,4 and 5: $C > A$ (12 votes)

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0						
B	1						
C							
D							

A v/s C.

- ▶ In column 1: $A > C$ (8 votes)
- ▶ In columns 2,3,4 and 5: $C > A$ (12 votes)
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	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0	0					
B	1						
C		1					
D							

A v/s C.

- ▶ In column 1: $A > C$ (8 votes)
- ▶ In columns 2,3,4 and 5: $C > A$ (12 votes)
- ▶ So A gets 0 points, C gets 1.

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0	0					
B	1						
C		1					
D							

A v/s D.

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0	0					
B	1						
C		1					
D							

A v/s D.

- ▶ In column 1: $A > D$ (8 votes)

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0	0					
B	1						
C		1					
D							

A v/s D.

- ▶ In column 1: $A > D$ (8 votes)
- ▶ In columns 2,3,4 and 5: $D > A$ (12 votes)

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0	0					
B	1						
C		1					
D							

A v/s D.

- ▶ In column 1: $A > D$ (8 votes)
- ▶ In columns 2,3,4 and 5: $D > A$ (12 votes)
- ▶ So A gets 0 points, D gets 1.

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0	0	0				
B	1						
C		1					
D			1				

A v/s D.

- ▶ In column 1: $A > D$ (8 votes)
- ▶ In columns 2,3,4 and 5: $D > A$ (12 votes)
- ▶ So A gets 0 points, D gets 1.

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0	0	0				
B	1						
C		1					
D			1				

B v/s C.

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0	0	0				
B	1						
C		1					
D			1				

B v/s C.

- ▶ In columns 1,4 and 5: $B > C$ (13 votes)

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0	0	0				
B	1						
C		1					
D			1				

B v/s C.

- ▶ In columns 1,4 and 5: $B > C$ (13 votes)
- ▶ In columns 2,3: $C > B$ (7 votes)

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0	0	0				
B	1						
C		1					
D			1				

B v/s C.

- ▶ In columns 1,4 and 5: $B > C$ (13 votes)
- ▶ In columns 2,3: $C > B$ (7 votes)
- ▶ So B gets 1 point, C gets 0.

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0	0	0				
B	1			1			
C		1		0			
D			1				

B v/s C.

- ▶ In columns 1,4 and 5: $B > C$ (13 votes)
- ▶ In columns 2,3: $C > B$ (7 votes)
- ▶ So B gets 1 point, C gets 0.

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0	0	0				
B	1			1			
C		1		0			
D			1				

B v/s D.

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0	0	0				
B	1			1			
C		1		0			
D			1				

B v/s D.

- ▶ In columns 1,4 and 5: $B > D$ (13 votes)

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0	0	0				
B	1			1			
C		1		0			
D			1				

B v/s D.

- ▶ In columns 1,4 and 5: $B > D$ (13 votes)
- ▶ In columns 2,3: $D > B$ (7 votes)

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0	0	0				
B	1			1			
C		1		0			
D			1				

B v/s D.

- ▶ In columns 1,4 and 5: $B > D$ (13 votes)
- ▶ In columns 2,3: $D > B$ (7 votes)
- ▶ So B gets 1 point, D gets 0.

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0	0	0				
B	1			1	1		
C		1		0			
D			1		0		

B v/s D.

- ▶ In columns 1,4 and 5: $B > D$ (13 votes)
- ▶ In columns 2,3: $D > B$ (7 votes)
- ▶ So B gets 1 point, D gets 0.

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0	0	0				
B	1			1	1		
C		1		0			
D			1		0		

C v/s D.

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0	0	0				
B	1			1	1		
C		1		0			
D			1		0		

C v/s D.

- ▶ In columns 1,2,3 and 5: $C > D$ (19 votes)

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0	0	0				
B	1			1	1		
C		1		0			
D			1		0		

C v/s D.

- ▶ In columns 1,2,3 and 5: $C > D$ (19 votes)
- ▶ In column 4: $D > C$ (1 vote)

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0	0	0				
B	1			1	1		
C		1		0			
D			1		0		

C v/s D.

- ▶ In columns 1,2,3 and 5: $C > D$ (19 votes)
- ▶ In column 4: $D > C$ (1 vote)
- ▶ So C gets 1 point, D gets 0.

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0	0	0				
B	1			1	1		
C		1		0		1	
D			1		0	0	

C v/s D.

- ▶ In columns 1,2,3 and 5: $C > D$ (19 votes)
- ▶ In column 4: $D > C$ (1 vote)
- ▶ So C gets 1 point, D gets 0.

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0	0	0				0
B	1			1	1		3
C		1		0		1	2
D			1		0	0	1

- ▶ Now we sum points up

	A-B	A-C	A-D	B-C	B-D	C-D	Total
A	0	0	0				0
B	1			1	1		3
C		1		0		1	2
D			1		0	0	1

- ▶ Now we sum points up
- ▶ and declare B the winner

The Pairwise Comparisons method has a built-in fairness criterion:

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Condorcet Criterion

If a candidate beats all the other candidates in head-to-head comparisons, then that candidate ought to be the winner.

It was first formulated in 1785 by



Marie Jean Antoine Nicolas de Caritat,
Marquis de Condorcet (1743-1794)

Remarks:

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- ▶ In Mathclub, B beats everyone else head-to-head.

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- ▶ Also, using the Pairwise Comparisons method, B wins.
- ▶ Hence this example satisfies the Condorcet criterion.
- ▶ By design, the **Pairwise Comparisons method always satisfies the Condorcet criterion.**

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- ▶ Plurality w/elimination (C won)

We now have all four voting methods and three out of four fairness criteria.

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Now let's motivate the last fairness criterion....

Example

- ▶ City Council election (A, B, C, D).

Example

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- ▶ Pairwise Comparisons.

Example

- ▶ City Council election (A, B, C, D).
- ▶ Pairwise Comparisons.
- ▶ 14 voters.

Preference schedule:

Choice	Number of ballots			
	5	6	1	2
1st	A	D	D	B
2nd	C	C	C	A
3rd	B	B	A	D
4th	D	A	B	C

Preference schedule:

Choice	Number of ballots			
	5	6	1	2
1st	A	D	D	B
2nd	C	C	C	A
3rd	B	B	A	D
4th	D	A	B	C

Find the winner of the election. [On the board.]

- ▶ After the election, it comes to light that candidate **C** was ineligible.

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- ▶ Since **C** lost anyway, nobody thinks this matters . . .

Except (losing) candidate **B**, who insists on a recount, with candidate **C** (being ineligible) removed from the ballots.

- ▶ When **C** is removed, everyone below **C** in the preference schedule moves up, but their relative positions and pairwise comparisons stay the same.

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- ▶ So we can use the same table with **C** removed.

	A-B	A-D	B-D	Total
A	0	1/2		1/2
B	1		1/2	3/2
D		1/2	1/2	1

- ▶ When **C** is removed, everyone below **C** in the preference schedule moves up, but their relative positions and pairwise comparisons stay the same.
- ▶ So we can use the same table with **C** removed.

	A-B	A-D	B-D	Total
A	0	1/2		1/2
B	1		1/2	3/2
D		1/2	1/2	1

Now the winner is **B**!

- ▶ We removed a loser from the ballot, and the winner changed.

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- ▶ This is a violation of the fourth fairness criterion:

Independence of Irrelevant Alternatives Criterion

If a non-winning candidate is removed from the ballot, and a recount is done, then the winner ought not to change.

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Colloquially this is the **losers don't matter** criterion.

Section 1.2.6 : Methods v/s Criteria

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1. Plurality

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1. Plurality
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voting methods:

1. Plurality
2. Borda count
3. Plurality with elimination

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voting methods:

1. Plurality
2. Borda count
3. Plurality with elimination
4. Pairwise comparisons

We also have 4 **fairness criteria**:

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1. Majority criterion

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1. Majority criterion
2. Monotonicity criterion

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2. Monotonicity criterion
3. Condorcet criterion

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1. Majority criterion
2. Monotonicity criterion
3. Condorcet criterion
4. Independence of Irrelevant Alternatives criterion

Remark

Each *one of the methods violates at least one of the fairness criteria.*

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- ▶ To say that a method violates a criterion **doesn't** mean that **every** time the method is used a violation will occur.

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Each *one of the methods* violates **at least one** *of the fairness criteria*.

- ▶ To say that a method violates a criterion **doesn't** mean that **every** time the method is used a violation will occur.
- ▶ It just means that there are **examples** when the method will violate the criterion.

Theorem (Arrow's impossibility theorem)

*For **any** voting method (not just these four) one can always find examples where at least one of the four fairness criteria is violated.*

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Remarks:

- ▶ This doesn't mean that all methods are bad, just none are perfect.

Theorem (Arrow's impossibility theorem)

*For **any** voting method (not just these four) one can always find examples where at least one of the four fairness criteria is violated.*

Remarks:

- ▶ This doesn't mean that all methods are bad, just none are perfect.
- ▶ It is mathematically impossible to devise a perfect voting system!

Here is a table recording which fairness criteria are satisfied/violated by the voting methods that we have studied. (Page 16 in the book.)

	Majority	Condorcet	Monotonicity	I.o.I.A.
Plurality	Satisfied	Violated	Satisfied	Violated
Borda Count	Violated	Violated	Satisfied	Violated
Plurality with Elimination	Satisfied	Violated	Violated	Violated
Pairwise Comparisons	Satisfied	Satisfied	Satisfied	Violated

Next time:

Section 1.2.7 : Ranking and Section
1.3 : One person-multiple votes, two
alternatives.