Section 3.2.4 continued: Sum/complement principles for probabilities.

Just like there is a product principle for probabilities, there is also a sum principle and a complement principle for probabilities.

Sum Principle for Probabilities

Suppose that an event can be broken down into **disjoint** parts ("sub-events"). Then the probability of the large event is the sum of the probabilities of the parts.

Example

Two distinct fair dice are rolled. What is the probability that the sum is a multiple of 3?

Solution

Since the sum is a number between 2 and 12, we can break down the event "sum is multiple of 3" into 4 cases

- *"sum=3"*
- *"sum=6"*
- *"sum=9"*
- *"sum=12"*

Solution

Note that these 4 cases are disjoint and they cover the whole event.

Solution

Note that these 4 cases are disjoint and they cover the whole event. Using the sum principle, we need to find the probability of each cases, and add them up.

There are a total of 36 outcomes in the sample space, and since the dice are fair, all outcomes are equally likely. Therefore the probability of each outcome is

$$\frac{1}{36}$$

There are a total of 36 outcomes in the sample space, and since the dice are fair, all outcomes are equally likely. Therefore the probability of each outcome is

$$\frac{1}{36}$$

All we need now is the number of outcomes in each event.

- "sum=3" has 2 outcomes ($1+2$ and $2+1$)
- "sum=6" has 5 outcomes
- "sum=9" has 4 outcomes
- "sum=12" has 1 outcome ($6+6$)

Thus we get

- $\mathrm{pr}(sum = 3) = \dfrac{2}{36}$

Thus we get

- $\mathrm{pr}(sum = 3) = \dfrac{2}{36}$
- $\mathrm{pr}(sum = 6) = \dfrac{5}{36}$

Thus we get

- $\mathrm{pr}(sum = 3) = \dfrac{2}{36}$
- $\mathrm{pr}(sum = 6) = \dfrac{5}{36}$
- $\mathrm{pr}(sum = 9) = \dfrac{4}{36}$

Thus we get

- $\mathrm{pr}(sum = 3) = \dfrac{2}{36}$
- $\mathrm{pr}(sum = 6) = \dfrac{5}{36}$
- $\mathrm{pr}(sum = 9) = \dfrac{4}{36}$
- $\mathrm{pr}(sum = 12) = \dfrac{1}{36}$

By the sum principle, we get that the probability of **"the sum is a multiple of 3"** is

$$\frac{2}{36}+\frac{5}{36}+\frac{4}{36}+\frac{1}{36}=\frac{12}{36}=\frac{1}{3}.$$

Example

Two distinct doctored dice

outcome	1	2	3	4	5	6
probability	0.35	0.1	0.1	0.1	0.1	0.25

are rolled. What is the probability that the sum is a multiple of 5?

Solution

Break down the event **E** *"sum is multiple of 5" into 2 disjoint cases*

- *"sum = 5"*
- *"sum = 10"*

Thus

$$pr(\mathbf{E}) = pr(\mathit{sum} = 5) + pr(\mathit{sum} = 10)$$

Thus

$$\mathrm{pr}(\mathsf{sum} = 5) =$$

$$0.1 \times 0.35 + 0.1 \times 0.1 + 0.1 \times 0.1 + 0.35 \times 0.1$$

$$= 0.09,$$

Thus

$$\mathrm{pr}(\mathsf{sum} = 5) =$$

$$0.1 \times 0.35 + 0.1 \times 0.1 + 0.1 \times 0.1 + 0.35 \times 0.1$$

$$= 0.09,$$

and

$$\mathrm{pr}(\mathsf{sum} = 10)$$

$$0.25 \times 0.1 + 0.1 \times 0.1 + 0.1 \times 0.25$$

$$= 0.06.$$

Now using the sum principle, the probability that the sum is a multiple of 5 is $0.09 + 0.06 = 0.15$.

Complement Principle for Probabilities

If an event E has probability $\mathrm{pr}(E) = p$, then its complementary event E' has probability $\mathrm{pr}(E') = (1 - p)$.

Example

Two distinct doctored dice are rolled. What is the probability that the sum is NOT a multiple of 5?

Solution

According to the complement principle

$$pr(\textit{sum is not a multiple of 5})$$
$$= 1 - pr(\textit{sum is a multiple of 5})$$
$$= 1 - 0.15 = 0.85 \textit{ or } 85\%.$$

Example

What is the probability that one of us (me and all students) have a birthday today?

Solution

- **Random Experiment:** *Pick a person.*

Solution

- **Random Experiment:** *Pick a person. Record* **S** *if he/she has a birthday today and* **F** *otherwise.*

Solution

- **Random Experiment:** *Pick a person. Record* **S** *if he/she has a birthday today and* **F** *otherwise.*
- $pr(\mathbf{S}) = \frac{1}{365} \qquad pr(\mathbf{F}) = \frac{364}{365}$

Solution

- **Random Experiment:** *Pick a person. Record* **S** *if he/she has a birthday today and* **F** *otherwise.*
- $pr(\mathbf{S}) = \frac{1}{365} \qquad pr(\mathbf{F}) = \frac{364}{365}$
- *Repeat this experiment 200 times and get a compound random experiment*

Solution

- *Then outcomes are strings*

$$\underbrace{\mathbf{FFFSFFFS}\ldots\mathbf{FFSF}}_{200}$$

Solution

- *Then outcomes are strings*

$$\underbrace{\mathbf{FFFSFFFS\ldots FFSF}}_{200}$$

- *The event* **E** *that we are interested in is:*

Solution

- *Then outcomes are strings*

$$\underbrace{\mathbf{FFFSFFFS\ldots FFSF}}_{200}$$

- *The event* **E** *that we are interested in is: there is at least one* **S** *in the outcome.*

Direct computation (slow and tedious):

Direct computation (slow and tedious):

E can be broken down into disjoint cases:

Direct computation (slow and tedious):

E can be broken down into disjoint cases:

- $\mathbf{E}_1 =$ there is exactly 1 **S** in the outcome

Direct computation (slow and tedious):

E can be broken down into disjoint cases:

- $\mathbf{E}_1 =$ there is exactly 1 **S** in the outcome
- $\mathbf{E}_2 =$
  there are exactly 2 **S**'s in the outcome

Direct computation (slow and tedious):

E can be broken down into disjoint cases:

- $\mathbf{E}_1 =$ there is exactly 1 **S** in the outcome
- $\mathbf{E}_2 =$
  there are exactly 2 **S**'s in the outcome
- $\mathbf{E}_3 =$
  there are exactly 3 **S**'s in the outcome
- $\vdots$

Direct computation (slow and tedious):

E can be broken down into disjoint cases:

- $\mathbf{E}_1 =$ there is exactly 1 **S** in the outcome
- $\mathbf{E}_2 =$ there are exactly 2 **S**'s in the outcome
- $\mathbf{E}_3 =$ there are exactly 3 **S**'s in the outcome
- $\vdots$
- $\mathbf{E}_{200} =$ there are exactly 200 **S**'s in the outcome

(Still the tedious way)

The probability of $\mathbf{E}_n$ is

$$\mathrm{pr}(\mathbf{E}_n) = {}_{200}C_n \times \left(\frac{1}{365}\right)^n \times \left(\frac{364}{365}\right)^{200-n}$$

(Still the tedious way)

So by the sum principle

$$\mathrm{pr}(\mathbf{E}) = \mathrm{pr}(\mathbf{E}_1) + \mathrm{pr}(\mathbf{E}_2) + \cdots + \mathrm{pr}(\mathbf{E}_{200})$$

$$= {}_{200}C_1 \times \left(\frac{1}{365}\right)^1 \times \left(\frac{364}{365}\right)^{199}$$

$$+ {}_{200}C_2 \times \left(\frac{1}{365}\right)^2 \times \left(\frac{364}{365}\right)^{198} + \cdots$$

$$+ {}_{200}C_{200} \times \left(\frac{1}{365}\right)^{200} \times \left(\frac{364}{365}\right)^0 = ???$$

Question

Do you really want to do this?

Question

Do you really want to do this?

Nope.

(Much better way)

Compute $\mathrm{pr}(\mathbf{E})$ using the complement principle

(Much better way)

Compute $\mathrm{pr}(\mathbf{E})$ using the
complement principle
The complement of **E** is

$$\mathbf{E}' = \text{“there is no } \mathbf{S} \text{ in the outcome”}$$

(Much better way)

Compute $\mathrm{pr}(\mathbf{E})$ using the
complement principle
The complement of $\mathbf{E}$ is

$$\mathbf{E}' = \text{“there is no } \mathbf{S} \text{ in the outcome”}$$

Then $\mathbf{E}'$ has only one outcome:
FFF. . . FF. Thus

$$\mathrm{pr}(\mathbf{E}') = \left(\frac{364}{365}\right)^{200}$$

(Much better way)

Compute $\mathrm{pr}(\mathbf{E})$ using the complement principle
The complement of **E** is

$$\mathbf{E}' = \text{“there is no } \mathbf{S} \text{ in the outcome”}$$

Then $\mathbf{E}'$ has only one outcome: **FFF. . . FF**. Thus

$$\mathrm{pr}(\mathbf{E}') = \left(\frac{364}{365}\right)^{200} \approx 0.58 = 58\%$$

Finally, by complement principle

$$\mathrm{pr}(\mathbf{E}) = 1 - \mathrm{pr}(\mathbf{E}') = 1 - 0.58 = 42\%$$

Last thing:

Sometimes we need to consider a sequence of independent random different experiments.

Example

Random Experiment: Toss a fair coin and throw a fair die.

Example

Random Experiment: Toss a fair coin and throw a fair die. Record **H** or **T**, and the number on the top of the die.

Example

Random Experiment: Toss a fair coin and throw a fair die. Record **H** or **T**, and the number on the top of the die.

a) How many outcomes are there in the probability space?

Example

b) Consider the event A given by: the coin was **H** and the number on the die is even.

Example

b) Consider the event A given by: the coin was **H** and the number on the die is even. How many outcomes are there in event A?

Example

b) Consider the event A given by: the coin was **H** and the number on the die is even. How many outcomes are there in event A?

c) What is the probability of A?

Solution

a) The outcomes are pairs

$$(H, 1), (T, 1), \ldots, (H, 6), (T, 6)$$

Solution

a) The outcomes are pairs

$$(H, 1), (T, 1), \ldots, (H, 6), (T, 6)$$

By product principle there are

$$2 \times 6 = 12$$

outcomes.

Solution

b) The outcomes in **A** *are*

$$(H,2),(H,4),(H,6)$$

Solution

b) The outcomes in **A** *are*

$$(H, 2), (H, 4), (H, 6)$$

(you still can use the product principle to count them: $1 \times 3 = 3$*)*

Solution

b) The outcomes in **A** *are*

$$(H,2),(H,4),(H,6)$$

(you still can use the product principle to count them: $1 \times 3 = 3$*)*

c) Since all outcomes are equally likely, the probability of **A** *is*

$$\frac{\textit{number of outcomes in } \mathbf{A}}{\textit{number of all possible outcomes}} = 25\%$$

End of Chapter 3.