Today's plan:

- We start Section 3.2: Probability
- Section 3.2.1: Random experiments; Probability spaces

Section 3.2.: Probability

Probability is a way to measure certainty/uncertainty.

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Example

Suppose we have a box full of marbles, red, blue and green.

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We have to pick one.

Example

- If we don’t know the number of marbles of each color, we can’t predict anything.

Example

- If we don't know the number of marbles of each color, we can't predict anything.
- If we know that most are green, we can *predict* that we'll pick a green one.

Example

- If we know that 90% of the marbles are green, we can predict with a 90% certainty that the one we pick will be green.

Section 3.2.1: Random experiments; probability spaces

Definition

A **random experiment** is an action with more than one possible **outcome** and whose outcome cannot be determined beforehand.

In the example above, picking a colored marble is a random experiment, because there are 3 possible outcomes (3 colors), but we don't know the outcome in advance.

Definition

Given a random experiment, the **Sample Space** is the set of all possible outcomes.

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In the example above, the sample space is {red, blue, green}.

Definition

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- each number is strictly between 0 and 1
- the sum of the numbers is 1

In the example above, if we know that 7% of the marbles are red and 3% are blue, the probability assignment is:

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outcome	red	blue	green
probability	0.07	0.03	0.9

Definition

A **Probability Space** is a sample space with a probability assignment.

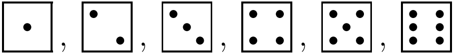
Definition

A fair die is a die in which all six sides are equally likely.

Definition

A fair die is a die in which all six sides are equally likely.
A doctored die is one where different sides might have different likelihoods.

When either die is rolled, the sample space has 6 outcomes:



However, the probability spaces will be different.

Example

Describe the probability space for the random experiment of rolling a fair die.

Solution

Since the die is fair, all six outcomes have the same probability.

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Their sum must be 1, and there are 6 outcomes, so each probability is $\frac{1}{6}$.

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Their sum must be 1, and there are 6 outcomes, so each probability is $\frac{1}{6}$.
The probability space is:

outcome	1	2	3	4	5	6
probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Example

Consider the random experiment of rolling a doctored die whose probability distribution is such that

- the probability of rolling a 6 is 0.25

Example

Consider the random experiment of rolling a doctored die whose probability distribution is such that

- the probability of rolling a 6 is 0.25
- the probability of rolling a 1 is 0.35

Example

Consider the random experiment of rolling a doctored die whose probability distribution is such that

- the probability of rolling a 6 is 0.25
- the probability of rolling a 1 is 0.35
- the probabilities of the other outcomes are equal

Example

Consider the random experiment of rolling a doctored die whose probability distribution is such that

- the probability of rolling a 6 is 0.25
- the probability of rolling a 1 is 0.35
- the probabilities of the other outcomes are equal

Describe the probability space for this random experiment.

Solution

The two outcomes "6" and "1" have a combined probability of

$$0.25 + 0.35 = 0.6$$

Solution

The two outcomes "6" and "1" have a combined probability of

$$0.25 + 0.35 = 0.6$$

Therefore the other 4 outcomes must have a combined probability of

$$1 - 0.6 = 0.4$$

Solution

Since these four have equal probabilities, each one is

$$\frac{0.4}{4} = 0.1$$

Solution

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Therefore, the probability space is

outcome	*1*	*2*	*3*	*4*	*5*	*6*
probability	0.35	0.1	0.1	0.1	0.1	0.25

Section 3.2.2: Events.

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We call a set of outcomes an **event**. The probability of an event is the sum of the probabilities of the outcomes in that event.

The probability of an event E is denoted by $pr(E)$.

Example

Consider the random experiment of rolling a fair die.

Example

Consider the random experiment of rolling a fair die. What are the probabilities of the following two events?

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- E_1: "roll a 1 or a 6"

Example

Consider the random experiment of rolling a fair die. What are the probabilities of the following two events?

- E_1: "roll a 1 or a 6"
- E_2: "roll a 2, a 3, a 4, or a 5"

Example

Consider the random experiment of rolling a fair die. What are the probabilities of the following two events?

- E_1: "roll a 1 or a 6"
- E_2: "roll a 2, a 3, a 4, or a 5"

Solution

We have

$$pr(E_1) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Solution

We have

$$pr(E_1) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

and

$$pr(E_2) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{2}{3}$$

Definition

An event may have no outcomes.

Definition

An event may have no outcomes. We call such an event the **empty event** or the **impossible event**.

Definition

An event may have no outcomes. We call such an event the **empty event** or the **impossible event**. An event that contains all possible outcomes is called a **certain event**.

Remarks:

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- An empty event always has probability 0, regardless of the probability distribution.

Remarks:

- An empty event always has probability 0, regardless of the probability distribution.
- A certain event always has probability 1, regardless of the probability distribution.

Example

The event

- **E**: "roll a 7" is an empty event

Example

The event

- **E**: “roll a 7” is an empty event

whereas the event

- **E**: “roll less than 7” is a certain event.

Section 3.2.3: Equally likely outcomes.

For some random experiments, all outcomes are equally likely.

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Since the sum of the probabilities of all outcomes must add up to 1, for a fair coin:

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- $pr(H) = 0.5$

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- $pr(H) = 0.5$
- $pr(T) = 0.5$

In general, if a random experiment has n possible outcomes, and they are all equally likely, then the probability of any one of the outcomes is $\frac{1}{n}$.

In general, if a random experiment has n possible outcomes, and they are all equally likely, then the probability of any one of the outcomes is $\frac{1}{n}$. The probability of an event consisting of k of the n possible outcomes is $\frac{k}{n}$.

Example

A card is drawn from a deck of 52 cards.

Example

A card is drawn from a deck of 52 cards.

- What is the probability of drawing the 10 of hearts?

Example

A card is drawn from a deck of 52 cards.

- What is the probability of drawing the 10 of hearts?
- What is the probability of drawing an ace?

Example

A card is drawn from a deck of 52 cards.

- What is the probability of drawing the 10 of hearts?
- What is the probability of drawing an ace?
- What is the probability of drawing a diamond?

Solution

All cards are equally likely.

Solution

All cards are equally likely.

- *There is one* 10 *of hearts in a deck.*

Solution

All cards are equally likely.

- *There is one* 10 *of hearts in a deck. Therefore, the probability of drawing the* 10 *of hearts is* $\frac{1}{52}$.

Solution

All cards are equally likely.

- *There is one 10 of hearts in a deck. Therefore, the probability of drawing the 10 of hearts is* $\frac{1}{52}$.
- *There are four aces in a deck.*

Solution

All cards are equally likely.

- *There is one 10 of hearts in a deck. Therefore, the probability of drawing the 10 of hearts is* $\frac{1}{52}$.
- *There are four aces in a deck. Therefore, the probability of drawing an ace is* $\frac{4}{52} = \frac{1}{13}$.

Solution

- *There are* 13 *diamonds in a deck.*

Solution

- *There are* 13 *diamonds in a deck. Therefore, the probability of drawing a diamond is* $\frac{13}{52} = \frac{1}{4}$.

The counting techniques from earlier are useful when all outcomes are equally likely.

The counting techniques from earlier are useful when all outcomes are equally likely. Figuring out the probability of an event amounts to two counting problems:

- Count the number **n** of possible outcomes for the random experiment.

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- Count the number **k** of outcomes in the event, say E, under consideration.

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- Count the number **k** of outcomes in the event, say E, under consideration.

Then

$$pr(E) = \frac{k}{n}$$

Example

A regular poker hand is drawn from a deck of cards.

Example

A regular poker hand is drawn from a deck of cards. What is the probability that it's a four-aces hand?

Example

A regular poker hand is drawn from a deck of cards. What is the probability that it's a four-aces hand? What is the probability it's a full house?

Solution

We computed before that there are

$${}_{52}C_5 = 2,598,960$$

different regular poker hands.

Solution

The number of four-aces hands is $48\ (= 52 - 4)$.

Solution

The number of four-aces hands is $48\ (= 52 - 4)$. *Therefore, the probability of getting a four-aces hand is*

$$\frac{48}{2,598,960} = 0.00001846892$$

Solution

The number of full houses is 3744.

Solution

The number of full houses is 3744. *Therefore, the probability of getting a full house is*

$$\frac{3744}{2,598,960} = 0.00144057623$$

Next time: Section 3.2.4: Iterated Two-Outcome Experiments