Section 3.1.5: Permutations and Combinations

When picking items from a set so that the order of the selection matters, we call this a **Permutation**.

When picking items from a set so that the order of the selection does not matter, we call this a **Combination**.

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How many different outcomes are there?

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How many different possibilities are there for the board?

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The number of different outcomes is denoted

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In general, if we want to select r items out of n, and the **order matters**, then the number of ways of doing that is denoted by

$$_nP_r$$

In Example (2), the order does not matter since the board members are equal. This is a **combination** problem.

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$$[A, B, C] \neq [A, C, B]$$

To write a **combination**, we use braces like {A,B,C}.

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$$\{A, B, C\} = \{A, C, B\}$$

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2. *Pick a Vice-president*
3. *Pick a Secretary*

Solution

Remarks:

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- *The selection of president will affect the choices for vice-president, since you can't be both.*
- *However, the* **number of choices** *for vice-president is* **independent** *of the selection of president.*

Solution

- *Number of choices for secretary is independent of the two previous steps.*

Solution

- *Number of choices for secretary is independent of the two previous steps.*
- *So we can apply the product principle to get the total number of outcomes.*

President		Vice-pres		Secretary		Slates
5	×	4	×	3	=	60

We see that

$$ {}_5P_3 = 5 \times 4 \times 3 $$

We go down by 1 since at each step we have 1 fewer item to pick from.

In general, if we want to pick r items out of n, and the **order matters**, the product principle tells us that

$$ {}_nP_r = \underbrace{n \times (n-1) \cdots \times (n-r+1)}_{r \text{ factors}} $$

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$$ {}_nP_n = n \times (n-1) \times (n-2) \cdots \times 2 \times 1 $$

We denote this number by $n!$, and call it **n factorial**.

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We can list all six:
[A,B,C], [A,C,B], [B,A,C], [B,C,A], [C,A,B], [C,B,A].

Using factorials we rewrite the formula for ${}_nP_r$ as:

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

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A club has an election for its executive board, consisting of 2 members (different than earlier example).There are 5 candidates, A, B, C, D and E. The top two will be on the board.How many outcomes are there?

Solution

We need to choose 2 board members out of 5 candidates. So the number of outcomes is

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But how do we compute that number?

Solution

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But outcomes like $[A, B]$ *and* $[B, A]$ *(which are different permutations) represent the same combination, namely* $\{A, B\}$.

Solution

So, in the above count of 20, each combination gets double counted. So, divide by 2, to get

$$ {}_5C_2 = \frac{5 \times 4}{2} = 10 $$

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$$ {}_rP_r = r! $$

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So, when we ignore the order, each combination gets counted $r!$ times.

Therefore the number of combinations of r items out of n is obtained by dividing the number of permutations by $r!$:

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or

$$ {}_nC_r = \frac{n \cdot (n-1) \cdots (n-r+1)}{1 \cdot 2 \cdot 3 \cdots r} $$

Example

$$_6P_3$$

Example

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$$ {}_6C_3 = \frac{{}_6P_3}{3!} = \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} $$

Example

$$ {}_6P_3 = 6 \cdot 5 \cdot 4 = 120 $$

$$ {}_6C_3 = \frac{{}_6P_3}{3!} = \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = 20 $$

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$$
\begin{aligned}
n! &= n \times (n-1) \cdots \times 2 \times 1 \\
0! &= 1
\end{aligned}
$$

Example

$$ {}_6P_0 = $$

Example

$$ {}_{6}P_{0} = \frac{6!}{(6-0)!} = 1 $$

Example

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$$ {}_6C_6 = $$

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$$ {}_{6}C_{6} = \frac{6!}{6! \cdot 0!} $$

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How many outcomes are there?

Solution

If 3 board members are chosen from a pool of 5 candidates, this can be done in

$$_5C_3 = \frac{5!}{2!\,3!} = \frac{120}{2 \cdot 6} = 10$$

different ways.

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a) How many different two-scoop orders are possible?

b) How many true double scoops are possible?

A true double is a two-scoop with 2 different flavors.

First consider the true doubles.

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$$ {}_7C_2 = \frac{7!}{2!5!} = \frac{5040}{2 \cdot 120} = 21 $$

ways to choose the two flavors.

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- pick the flavors (21 choices)

By the product principle there are

$$3 \times 21 = 63$$

different true doubles.

For the first part of the problem (number of two-scoop orders, period), we can break it down into two disjoint cases.

- we have a true double
- both scoops have the same flavor

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- we have a true double (63 orders)
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- we have a true double (63 orders)
- both scoops have the same flavor (21 orders)

The first case is already covered (63 orders).

For the second case, we need to pick

- a container (3 choices)
- a single flavor (7 choices)

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- a container (3 choices)
- a single flavor (7 choices)

so there are $3 \times 7 = 21$ two-scoop of a single flavor.

Combining the two cases with the sum principle, there are a total of

$$63 + 21 = 84$$

two-scoop ice cream orders.

Next time: Counting poker hands.