Section 3.1.3.: Sum principle.

Sum Principle:

IF a counting task can be broken down into a sequence of disjoint cases

Sum Principle:

IF a counting task can be broken down into a sequence of disjoint cases

THEN the number of outcomes is the sum of the numbers of outcomes for the cases.

Example

We are planning to drive from Santa Rosa to Carbondale, eating along the way. There are three different choices of roads.

Example

- On the first road there are four restaurants.
- On the second road there are two restaurants.
- On the third road there is one restaurant.

How many alternatives do we have for our trip?

Solution

We can break this into three disjoint cases, one per road.

Solution

We can break this into three ***disjoint*** *cases, one per road.*

- *For the first road there are 4 choices of restaurant.*

Solution

We can break this into three disjoint cases, one per road.

- *For the first road there are 4 choices of restaurant.*
- *For the second road there are 2 choices.*

Solution

We can break this into three disjoint cases, one per road.

- *For the first road there are 4 choices of restaurant.*
- *For the second road there are 2 choices.*
- *For the third road there is just 1 choice.*

Solution

The three cases are mutually disjoint, i.e., no two of them can happen at the same time.

Solution

The three cases are mutually disjoint, i.e., no two of them can happen at the same time.
According to the sum principle, there are

$$4 + 2 + 1 = 7$$

alternatives for the trip.

Road 1		Road 2		Road 3		Total
4	+	2	+	1	=	7

Sometimes one needs to combine both the product principle and the sum principle.

Example

The friends will take one of 3 cars, and spend the night at a motel in Carbondale.

Example

The friends will take one of 3 cars, and spend the night at a motel in Carbondale. There are still 3 different roads....

Example

- On the first road there are 4 restaurants.

Example

- On the first road there are 4 restaurants.
- On the second road there are 2 restaurants.

Example

- On the first road there are 4 restaurants.
- On the second road there are 2 restaurants.
- On the third road there is 1 restaurant.

Example

- On the first road there are 4 restaurants.
- On the second road there are 2 restaurants.
- On the third road there is 1 restaurant.

There are 5 motels.

Example

- On the first road there are 4 restaurants.
- On the second road there are 2 restaurants.
- On the third road there is 1 restaurant.

There are 5 motels. How many alternatives do they have for their trip?

Solution

Picking an alternative involves some steps:

1. *Pick a car (3).*

Solution

Picking an alternative involves some steps:

1. *Pick a car (3).*
2. *Pick a road (3).*

Solution

Picking an alternative involves some steps:

1. *Pick a car (3).*
2. *Pick a road (3).*
3. *Pick a place to eat on that road (depends).*

Solution

Picking an alternative involves some steps:

1. *Pick a car (3).*
2. *Pick a road (3).*
3. *Pick a place to eat on that road (depends).*
4. *Pick a motel (5).*

Solution

The number of restaurants depends on the choice in step 2, so we can't apply the product principle right away.

Solution

The number of restaurants depends on the choice in step 2, so we can't apply the product principle right away. But, if we combine steps 2 and 3 into one step, we will have 3 independent steps.

Solution

1. *Pick a car (3).*

Solution

1. *Pick a car (3).*
2. *Pick a road and a restaurant (use sum principle).*

Solution

1. *Pick a car (3).*
2. *Pick a road and a restaurant (use sum principle).*
3. *Pick a motel (5).*

Now we can apply the product principle, since these steps are independent. For step 2 we've already seen there are 7 outcomes.

$$\underset{3}{\boxed{\text{Car}}} \times \underset{\underbrace{(4+2+1)}_{7}}{\boxed{\text{Road/ Restaurant}}} \times \underset{5}{\boxed{\text{Motel}}} = \underset{105}{\boxed{\text{Trip}}}$$

We combined the sum principle with the product principle.

We combined the sum principle with the product principle.

- The sum principle appeared in a step of the product principle.

We combined the sum principle with the product principle.

- The sum principle appeared in a step of the product principle.
- Sometimes the product principle can appear in a case of the sum principle.

We combined the sum principle with the product principle.

- The sum principle appeared in a step of the product principle.
- Sometimes the product principle can appear in a case of the sum principle.
- Sometimes the interaction can get even more complex.

Example

Freshmen used to be assigned login IDs consisting of two letters (lowercase) and four digits. For example, **bg1352**.

Example

Now the login IDs may also consist of 5 letters, upper and lowercase, and they are case sensitive. So **ipods** is different from **iPods**.

Example

Now the login IDs may also consist of 5 letters, upper and lowercase, and they are case sensitive. So **ipods** is different from **iPods**.

How many different login IDs are possible when these two schemes are combined?

Solution

The two schemes are disjoint, so we can apply the sum principle.

Solution

The two schemes are disjoint, so we can apply the sum principle.

- *Using the product principle with the old scheme (like* **bg1352***) we get*

$$26 \times 26 \times 10^4 = 6,760,000$$

login ids.

Solution

- *With the new scheme (like* **iPods***)there are*

$$52^5 = 380,204,032$$

login ids.

Solution

- *With the new scheme (like* **iPods***)there are*

$$52^5 = 380,204,032$$

login ids.

Solution

Now using the sum principle, combined there are a total of

$$6,760,000 + 380,204,032 = 386,964,032$$

login ids.

Sometimes a counting task can be broken down into several cases, but the cases are not disjoint, and therefore we cannot use the sum principle.

Example

A club consisting of five members, Ann, Bert, Charles, Doris, and Ernest, elects a president and a secretary.

Example

A club consisting of five members, Ann, Bert, Charles, Doris, and Ernest, elects a president and a secretary. The club bylaws mandate at least one of the officers must be a woman.

Example

A club consisting of five members, Ann, Bert, Charles, Doris, and Ernest, elects a president and a secretary. The club bylaws mandate at least one of the officers must be a woman. In how many ways can both officers be elected?

Solution

Since one of the officers must be a woman, we can break down the task into two cases. Either:

Solution

Since one of the officers must be a woman, we can break down the task into two cases. Either:

1. *The president is a woman (2 choices), and the secretary is any of the remaining members (4 choices).*

Solution

Since one of the officers must be a woman, we can break down the task into two cases. Either:

1. *The president is a woman (2 choices), and the secretary is any of the remaining members (4 choices). (Here there are 8 options, by the product principle).*

Solution

Or:

Solution

Or:

2. *The secretary is a woman (2 choices), and the president is any of the remaining members (4 choices).*

Solution

Or:

2. *The secretary is a woman (2 choices), and the president is any of the remaining members (4 choices). (Here there are 8 options, by the product principle).*

We might be tempted to use the sum principle and conclude that there are

$$8 + 8 = 16$$

ways to select president and secretary with at least one of them being a woman.

However, the sum principle doesn't apply, since the two cases are not disjoint.

However, the sum principle doesn't apply, since the two cases are not disjoint.
For example, having Ann as president and Doris as secretary fits into both cases. Mathematically, the pair [*P:Ann - S:Doris*] got double counted.

However, the sum principle doesn't apply, since the two cases are not disjoint.
For example, having Ann as president and Doris as secretary fits into both cases. Mathematically, the pair [*P:Ann - S:Doris*] got double counted. (Similarly, the pair [*P:Doris - S:Ann*] got double counted).

It turns out these are the only double-counted outcomes, and so there are 14 (not 16) different ways the election can go.

Section 3.1.4: Complement Principle

Sometimes a counting task is the exact opposite of another counting task, and counting the number of outcomes might be more easily done by...

- first counting the number of ways in which its opposite can be done

- first counting the number of ways in which its opposite can be done
- then subtracting this number from a total

Example (same example)

A club consisting of five members, Ann, Bert, Charles, Doris, and Ernest, is going to elect a president and a secretary. The club bylaws mandate that at least one of the officers must be a woman. In how many ways can both officers be selected?

Solution

First note that using the product principle, there are

$$5 \times 4 = 20$$

different ways to select president and secretary with no restrictions on gender.

Solution

At least one of the officers must be a woman....

Solution

At least one of the officers must be a woman.... The opposite of this restriction would be that both officers are men.

Solution

There are $3 \times 2 = 6$ *different ways to select president and secretary with both of them being men.*

Solution

There are $3 \times 2 = 6$ *different ways to select president and secretary with both of them being men.*

Using the complement principle, there are

$$20 - 6 = 14$$

different ways to select president and secretary with at least one of them being a woman.

In order to apply the complement principle, our set must be part of a larger set, and we must be able to count both the complement and the larger set.

Complement Principle

When a counting task T is a restriction of another task S, the number of ways of doing T is equal to the number of ways of doing S minus the number of ways of doing T', the complement of T.