Chapter 3: Counting and Probability

Motivation

- Probability studies the chances of events happening

For instance, the probability of:

- winning the lottery

For instance, the probability of:

- winning the lottery
- stock prices going up

For instance, the probability of:

- winning the lottery
- stock prices going up
- a coincidence happening

- Counting is the basis for probability

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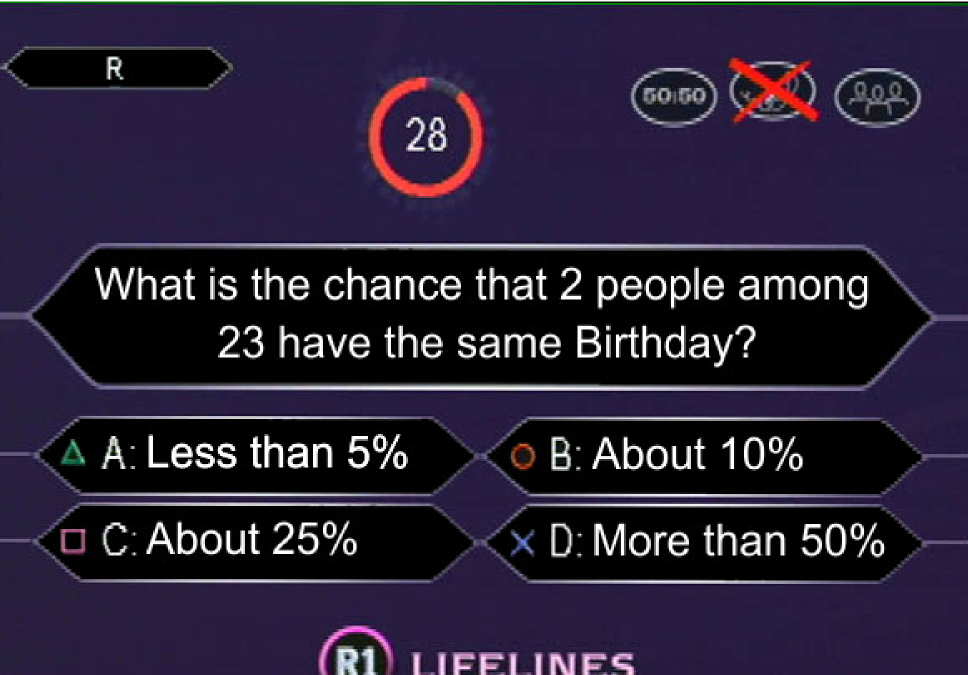
- Probability is the basis for statistics (Chapter 4)

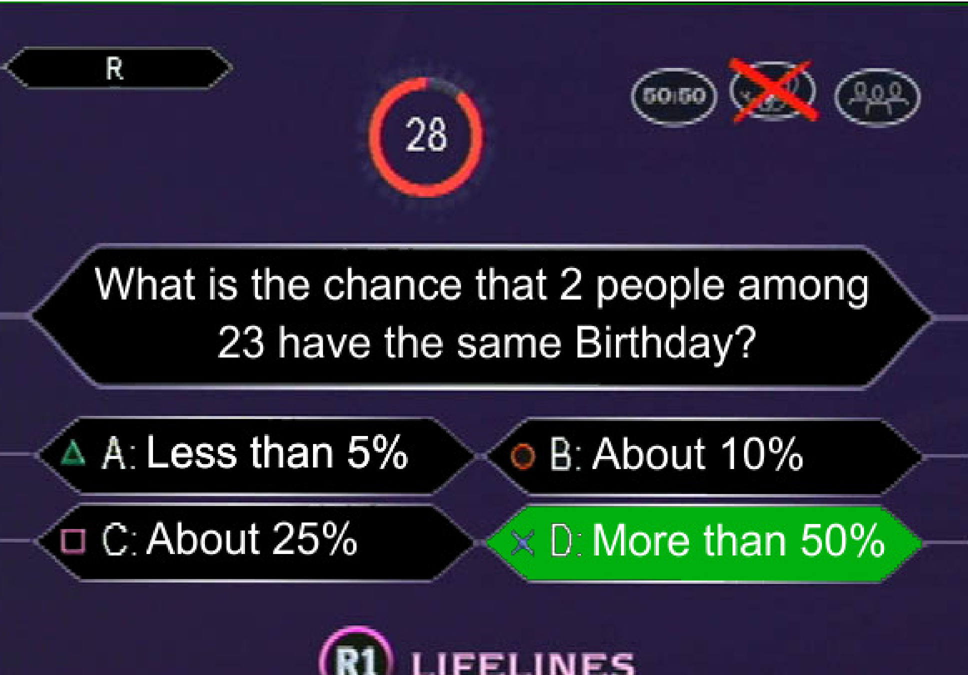
Some history

The first people who studied probability were French mathematicians Blaise Pascal and Pierre de Fermat (17th century)

Pierre de Fermat (1601 - 1665)

Blaise Pascal (1623 - 1662)


R
28
50:50
What is the chance that 2 people among 23 have the same Birthday?
A: Less than 5%
B: About 10%
C: About 25%
D: More than 50%
R1 LIFELINES


R
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50:50
What is the chance that 2 people among 23 have the same Birthday?
A: Less than 5%
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R1 LIFELINES

In contrast, you need 253 people before the probability is 1/2 that someone has a specific birth date, like July 4

Raise your hand if your birthday is:

- July 4

Raise your hand if your birthday is:

- July 4
- March 4

Raise your hand if your birthday is:

- July 4
- March 4
- February 7

Raise your hand if your birthday is:

- July 4
- March 4
- February 7
- December 25

Raise your hand if your birthday is:

- July 4
- March 4
- February 7
- December 25
- November 14

Raise your hand if your birthday is:

- July 4
- March 4
- February 7
- December 25
- November 14
- December 15

Raise your hand if your birthday is:

- July 4
- March 4
- February 7
- December 25
- November 14
- December 15

Raise your hand if your birthday is:

- July 4
- March 4
- February 7
- December 25
- November 14
- December 15

In theory, we should get about 2 or 3 "coincidences"

Section 3.1: Counting Principles

We consider **counting** problems like:

"In how many ways can . . . be done?".

To do this, we need some counting principles.

Section 3.1.1: List and Count (brute force)

Sometimes you just list all the possible ways and then count them.

Example

- Flip a coin, how many outcomes?

Example

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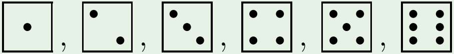
  Answer: 2 possible outcomes: Heads or Tails.

Example

- Flip a coin, how many outcomes?
  Answer: 2 possible outcomes: Heads or Tails.
- Roll a normal die, how many outcomes?

Example

- Flip a coin, how many outcomes?
  Answer: 2 possible outcomes: Heads or Tails.
- Roll a normal die, how many outcomes?
  Answer: 6 possible outcomes:



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In general, we need some elementary counting principles that make things easier.

These principles include:

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Product principle: when the problem can be broken down into independent steps.

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Sum Principle: when the problem can be broken down into disjoint cases.

Complement Principle: when the problem is the exact opposite of another one.

Section 3.1.2: Product principle

The product principle or multiplication rule states:

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IF a counting task can be broken down into a sequence of independent steps, i.e., later steps don't depend on choices from previous steps...

THEN the answer is the product of the answers for the individual steps.

Example

Roll two (distinguishable) dice. How many different combinations can come up?

Solution

The problem can be broken down into two steps:

Solution

The problem can be broken down into two steps:

- *Count the number of outcomes on the first die.*
  There are 6.

Solution

The problem can be broken down into two steps:

- *Count the number of outcomes on the first die.*
  There are 6.
- *Count the number of outcomes on the second die.*
  There are also 6.

Solution

These steps are independent, so

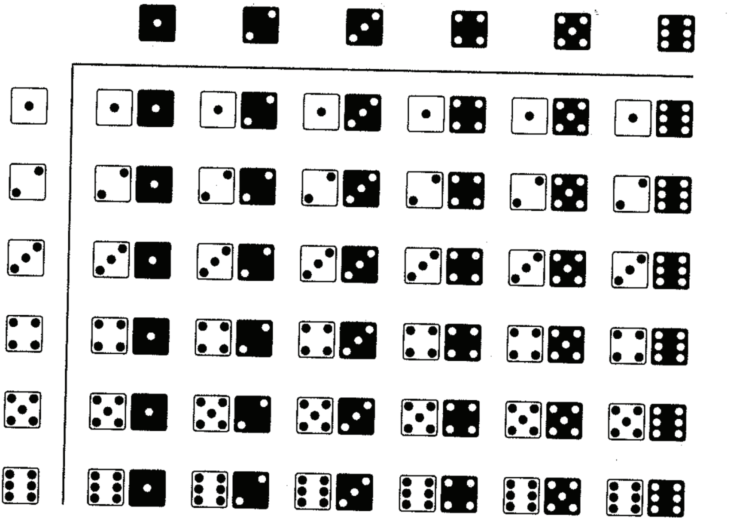
Solution

These steps are independent, so by the product principle there are

$$6 \times 6 = 36$$

different combinations.

Figure 1. Throwing a pair of dice. There are 36 ways for the dice to fall, shown in the body of the diagram; all are equally likely.

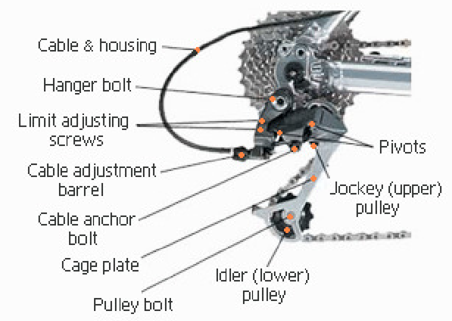


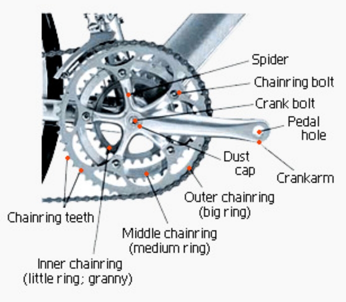
Example

Why do bikes often have 18 different gears? Why not 19?

Example

Why do bikes often have 18 different gears? Why not 19?





Solution

Counting combinations of front and back gears can be broken down into two steps:

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- *Count the number of rear gears. For example there could be 6.*

Solution

Counting combinations of front and back gears can be broken down into two steps:

- *Count the number of rear gears. For example there could be 6.*
- *Count the number of front gears. For example there could be 3.*

Solution

By the product principle there are

$$6 \times 3 = 18$$

different gears. (And we only needed 6 in back and 3 in front.)

To get 19 we'd need $1 \times 19 = 19$, which is inconvenient.

Example

An ice cream parlor has 7 different flavors. They serve either on a regular cone, sugar cone, or a dish. How many different single-scoop ice cream orders are possible?

Solution

This can be broken down into two steps:

- *Pick a container.*
  Can be done in 3 different ways: regular, sugar, dish.

Solution

This can be broken down into two steps:

- *Pick a container.*
  Can be done in 3 different ways: regular, sugar, dish.
- *Pick a flavor.*
  Can be done in 7 different ways, namely the 7 different flavors.

Solution

These are independent choices, so the product principle says there are

$$7 \times 3 = 21$$

different single-scoop ice cream orders.

Cont.		Flavor		Order
3	×	7	=	21

The number of steps can be more than 2. But the product principle still applies as long as the steps are independent of each other, just like how the choice of the flavor didn't depend on the choice of container.

Example

Maria has 3 pairs of shoes, 2 skirts, 4 blouses and 1 jacket. How many different outfits can she assemble? (Assume an "outfit" is a pair of shoes, a skirt, and a blouse, with the jacket optional).

Solution

We can view this as a 4-step process:

Solution

1. *Pick a pair of shoes. There are 3 choices.*

Solution

1. *Pick a pair of shoes. There are 3 choices.*
2. *Pick a skirt. There are 2 choices.*

Solution

1. *Pick a pair of shoes. There are 3 choices.*
2. *Pick a skirt. There are 2 choices.*
3. *Pick a blouse. There are 4 choices.*

Solution

1. *Pick a pair of shoes. There are 3 choices.*
2. *Pick a skirt. There are 2 choices.*
3. *Pick a blouse. There are 4 choices.*
4. *Finally, decide on the jacket. There are 2 choices (to wear or not to wear)*

Solution

By the product principle, this can be done in $3 \times 2 \times 4 \times 2 = 48$ *different ways. Therefore 48 different outfits can be assembled.*

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Shoes		Skirt		Blouse		Jacket		Outfit
3	$\times$	2	$\times$	4	$\times$	2	$=$	48

When the different steps are not independent of each other, the product principle may not apply.

Example

We are planning to drive from Santa Rosa to Carbondale, eating lunch along the way.

Example

We are planning to drive from Santa Rosa to Carbondale, eating lunch along the way. To go from Santa Rosa to Carbondale there are three different roads.

Example

- On the first road there are four restaurants.

Example

- On the first road there are four restaurants.
- On the second road there are two restaurants.

Example

- On the first road there are four restaurants.
- On the second road there are two restaurants.
- On the third road there is one restaurant.

Example

- On the first road there are four restaurants.
- On the second road there are two restaurants.
- On the third road there is one restaurant.

How many alternatives do we have for our trip?

This breaks down naturally into two steps:

- pick a road

This breaks down naturally into two steps:

- pick a road
- pick a place for lunch

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- pick a road
- pick a place for lunch

but the second step depends on the choice made for the first one!

More importantly, the **number of choices** in the second step depends on the choice made in step 1.

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We can't apply the product principle.

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We can't apply the product principle.

We'll see later we can use the sum principle.

Sometimes the number of choices in a step doesn't depend on the previous steps, even though the choices themselves may depend on earlier choices.

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Example

Election for President and Secretary.

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Four candidates: A, B, C, and D.

Example

Election for President and Secretary.
Four candidates: A, B, C, and D.
How many possible outcomes are there?

Solution

We can list the outcomes by picking one candidate for president, and then one of the remaining candidates for secretary.

Solution

Note that the second step does depend on the choice for the first step.

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- *If B is chosen as president then the only choices for secretary are A, C and D.*

Solution

Note that the second step does depend on the choice for the first step.

- *If B is chosen as president then the only choices for secretary are A, C and D.*
- *But if D is chosen for president then B is still a choice for secretary.*

Solution

However, regardless of the choice for president, there are always 3 choices for secretary.

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However, regardless of the choice for president, there are always 3 choices for secretary. The number of choices for step 2 is independent of step 1.

Solution

However, regardless of the choice for president, there are always 3 choices for secretary. The number of choices for step 2 is independent of step 1. The product principle applies:

President		Secretary		Outcomes
4	$\times$	3	$=$	12

Product Principle

When a counting task can be broken down into steps so that the number of alternatives in each step is independent of the previous steps, then the number of ways the counting task can be done is the product of the number of ways the individual steps can be done.