Today's plan:

- Section 2.4.1 continued. (Continuous–different values–equal rights)

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Question

How many cake pieces will be there with n parties?

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With n players there will be

$$1$$

Question

How many cake pieces will be there with n parties?

With n players there will be

$$1 \cdot 2$$

Question

How many cake pieces will be there with n parties?

With n players there will be

$$1 \cdot 2 \cdot 3$$

Question

How many cake pieces will be there with n parties?

With n players there will be

$$1 \cdot 2 \cdot 3 \cdots n$$

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For example, a fair division of the cake among just 8 people by the Cut and Choose-method will require

$$8! = 40320 \quad \text{pieces}$$

Question (Real)

- *How to reduce the number of pieces?*
- *How to make a division envy-free?*

Envy-free division was first solved for the 3 player case in 1960, independently by John Selfridge (NIU) and John Horton Conway (Cambridge). The best algorithm uses only 5 cuts.

Envy-free division for 4 or more players was a difficult open problem of the twentieth century. It was finally solved by Steven Brams and Alan Taylor in 1995.

[End of Cut and Choose method]

Claim and Challenge Method

When the number of players is large, the Claim and Challenge method is more appropriate than the Cut and Choose method.

Claim and Challenge Method

- Randomly select the order of the players.
- In each round the first player claims a piece of the cake.
- The other players take turns to either challenge the claim, or pass.

Claim and Challenge Method

- If a player challenges a claim, the challenger reduces the size of the claimed piece, and becomes the claimant of that reduced piece.

Claim and Challenge Method

- At the end of the round, the claimant that went unchallenged gets his claimed piece, and is out of the process.
- The rounds are repeated until all the players get their piece. It takes one fewer round than the number of players.

Example

Five heirs to an estate decide to use the Claim and Challenge method to divide up a parcel of land.

Solution

We will call the five heirs (players) P1, P2, P3, P4, and P5.

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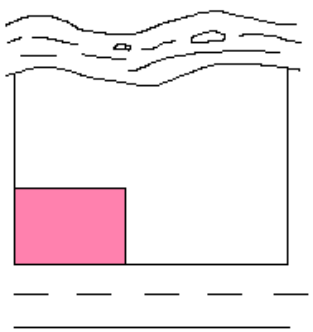
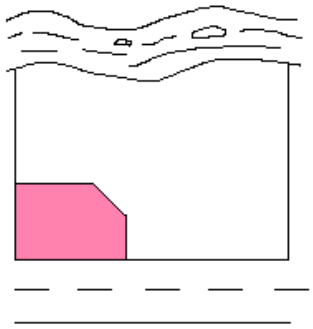
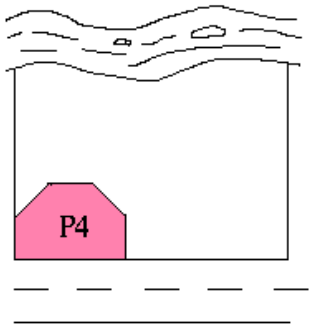
- *It takes four rounds of Claim and Challenge to divide the land.*
- *At the end of each round, one of the heirs will have a piece of land he/she considers a fair share.*



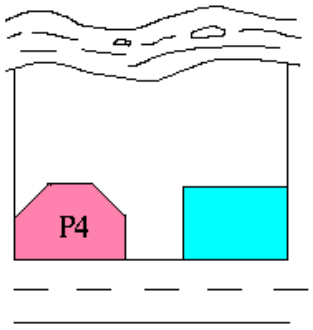
Round 1: P1 claims



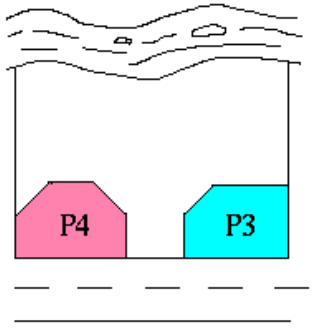
P2 passes; P3 challanges



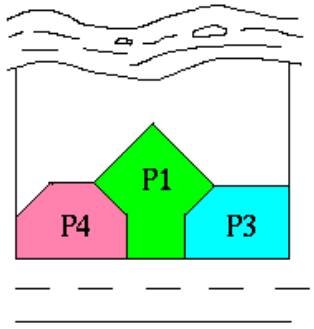
P4 challanges ; P5 passes
P4 keeps the piece



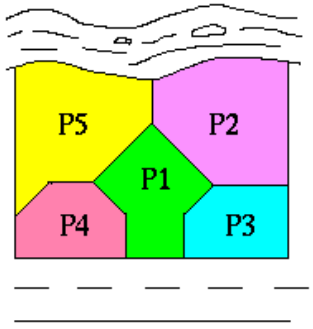
Round 2: P1 claims



P2 passes; P3 challanges
P5 passes. P3 keeps it.



Round 3: P1 claims
P2 and P5 pass;
P1 keeps the piece



In the last round, P2 claims,
P5 passes.

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- It is to the claimant's advantage not to claim more than what he considers fair. Claiming too much begs for a challenge.
- It is possible, and probably convenient, to randomly rearrange the remaining players after each round. We didn't do that in this example.

- The last round, between two players, can be done by the *you-cut-I-choose* method, if the players so wish.

Example

Five friends, P_1, P_2, P_3, P_4, and P_5, buy a pizza.

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Five friends, P_1, P_2, P_3, P_4, and P_5, buy a pizza. The Cut and Choose method would result in 120 slices. Too many! They decide to use the Claim and Challenge method instead.

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  One less than the number of players, so 4 rounds
- Logic problem: Given that P_2 starts the second round, who got his/her piece in the first round?
  Since P_1 is no longer around, she/he got her/his piece of pizza in the first round.

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  P_1 claims, P_2, P_3, P_4, and P_5 pass
- Who are the players in the second round?
  The players of the second round are P_2, P_3, P_4, and P_5.

- Given that in the second round every claim is challenged, who gets a piece of pizza at the end of the second round?

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  Everyone challenged so P_5 got pizza in round 2.

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  The players in Round 3 are P_2, P_3, and P_4
- Given that P_3 gets pizza in Round 3, describe the third round in terms of claims, challenges, and passes.
  P_2 claims, P_3 challenges, and P_4 passes.

- Who are the players in the last round? Given that they use the you-cut-I-choose method for this round, who will cut and who will choose? Who is the last player to receive pizza?

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  The players of the last round are P_2 and P_4. P_2 cuts and P_4 chooses; P_2 takes what is left. P_2 is the last player to receive pizza.