

## Math 605: Final Exam 2017 Spring

Send an electronic version to [zshang@binghamton.edu](mailto:zshang@binghamton.edu) before 5pm May 15, 2017. Late submission will NOT be counted.

- (1) (5 points) Suppose that  $Y \sim N(\theta, I_n)$ . Show that  $\hat{\theta} = Y$  is a minimax estimator of  $\theta$ .
- (2) (5 points) Show that the positive-part James-Stein estimator has smaller estimation risk than the James-Stein estimator. Find the ratio of these risks as the number of observations diverges.
- (3) Suppose  $X|\theta \sim N(\theta, I_p)$ , where  $\theta \in \mathbb{R}^p$  is unknown mean vector. The aim is to estimate  $\theta$ . Consider the following general form of James-Stein estimator

$$\hat{\theta}_a = \left(1 - \frac{a}{\|X\|_2^2}\right) X, \quad 0 < a < 2(p-2).$$

- (a) (3 points) Show that  $E_\theta \|\hat{\theta}_a - \theta\|_2^2 < E_\theta \|X - \theta\|_2^2$  for any  $\theta$ .
- (b) Consider a linear regression  $Y_i = X_i\beta + \epsilon_i$ ,  $i = 1, \dots, n$ , where  $\beta \in \mathbb{R}^p$  with  $p$  fixed and  $\epsilon_i \stackrel{iid}{\sim} N(0, 1)$ .
  - (b1) (2 points) Derive a James-Stein estimator  $\hat{\beta}^{JS}$  for  $\beta$ .
  - (b2) (4 points) Show that  $\hat{\beta}^{JS}$  dominates the ordinary least square estimator in terms of  $L^2$  risk under any  $\beta \in \mathbb{R}^p$ .
  - (b3) (4 points) Can you further improve  $\hat{\beta}^{JS}$ ?
- (4) Consider again the Bayesian variable selection problem with the second type of spike & slab prior, i.e., the following hierarchical model

$$Y_i | X_i, \beta_1, \dots, \beta_p, \sigma^2 \stackrel{iid}{\sim} N(\beta_1 X_{i1} + \dots + \beta_p X_{ip}, \sigma^2), \quad i = 1, \dots, n,$$

with prior distributions

$$\begin{aligned} \beta_j | \gamma_j, \sigma^2 &\stackrel{iid}{\sim} (1 - \gamma_j)\delta_0 + \gamma_j N(0, c\sigma^2) \\ \gamma_1, \dots, \gamma_p &\stackrel{iid}{\sim} \text{Bernoulli}(p) \\ \sigma^2, \tau_1^2, \dots, \tau_p^2 &\stackrel{iid}{\sim} IG(a, b), \end{aligned}$$

where  $a, b, c > 0, 0 < p < 1$  are hyperparameters.

- (a) (4 points) Derive (in full details) an empirical Bayes approach for estimating  $\beta_1, \dots, \beta_p$ .
- (b) (3 points) Examine your EB approach via the pseudo data provided in my website. Show full details.

(5) (10 points) Approximate the following integral  $I(\gamma)$  as  $\gamma \rightarrow \infty$ :

$$I(\gamma) = \int_{x_1, \dots, x_n \in S^1} \exp \left( \gamma \sum_{i,j=1}^n a_{ij} x_i^T x_j \right) dx_1 \cdots dx_n,$$

where  $S^1$  is the unit circle and  $A = [a_{ij}]_{i,j=1}^n$  is any  $n \times n$  symmetric real matrix.