Math 447 Spring 2018
Final Exam

1. (a) (4 points) You are given $\mathbb{P}[A \cup B] = 0.7$ and $\mathbb{P}[A \cup B^c] = 0.9$, where B^c denote the complement of B.

   Calculate $\mathbb{P}[A]$.

   (b) (4 points) You are given following information:

   - At a cafe, all customers order at least one item.
   - 30% of the customers order more than one item.
   - 60% of the customers order a coffee.
   - Of those customers who order more than one item, 85% order a coffee.

   Calculate the probability that a randomly selected customer orders exactly one item and this item is not a coffee.

   (c) (4 points) For customers that shop at an online store it is known that:

   - A customer is twice as likely to purchase a music album as a movie.
   - The event that a customer purchases a music album is independent of the event that he or she purchases a movie.
   - The probability that a customer purchases both a music album and a movie is 0.15.

   Calculate the probability that a customer purchases neither a music album nor a movie.

2. (a) (4 points) An electronic system contains three components that operate independently. The probability of each component's failure is 0.05. The system will break down if and only if at least two components fail.

   Calculate the probability that the system will overheat.

   (b) (4 points) In a sample of 20 X-phones, 7 X-phones are damaged. The X-phones are randomly inspected, one at a time, without replacement, until the fourth damaged X-phone is discovered. Calculate the probability that exactly 12 X-phones are inspected.

3. The weight of each box that the post office receives is an independent random variable with mean 4 and variance 2.

   (a) (5 points) Use Tchebysheff's inequality to estimate the probability the total weight of 50 boxes is between 180 and 220.

   (b) (5 points) Furthermore, assume the distribution of each box is normal. Compute the the probability the total weight of 50 boxes is between 180 and 220.

4. Let the random variable T have the gamma distribution with parameters $\alpha = 2$ and $\beta = 1/2$.

   Conditioned on the event $T = t$, random variable X is uniformly distributed on $[0, t]$.

   (a) (3 points) Compute the joint pdf of X and T.

   (b) (3 points) Compute the marginal pdf of X.

   (c) (2 points) What is the conditional pdf of T given $X = x$?

   (d) (2 points) Calculate conditional expectation $\mathbb{E}(T|X = x)$.

5. (10 points) A health study tracked a group of people for five years. At the beginning of the study, 10 percent were classified as heavy smokers, 30 percent as light smokers, and 60 percent as nonsmokers.

   Results of the study showed that light smokers were three times as likely as nonsmokers to die during the five-year study, but only half as likely as heavy smokers.

   A randomly selected participant from the study died over the five-year period. Calculate the probability that the participant was a heavy smoker.

6. A random variable X has probability density function

$$f(x) = c(3x^2 - 30x + 80), \quad 0 < x < 10,$$

   and 0, elsewhere, where c is a constant.

   (a) (3 points) Calculate c.

   (b) (3 points) Calculate $\mathbb{P}(X > 5)$.

   (c) (4 points) Calculate $\mathbb{P}(X < 8 | X > 5)$.

7. The sales in departments A and B are given by random variables M and N, respectively. The data analysis reveals that $\text{Var}(M) = 1600$, $\text{Var}(N) = 900$, and the correlation, ρ, between M and N is 0.64.

   (a) (5 points) Calculate the covariance between M and N.

   (b) (5 points) Calculate the variance of $M - N$.

8. The random variables X and Y are independently and identically distributed. The moment generating function of each random variable is

$$m(t) = \frac{1}{(1 - 1.5t)^2},$$

   (a) (2 points) What is the moment generating function of $X + Y$?

   (b) (3 points) Calculate $\mathbb{E}(X + Y)$.

   (c) (3 points) Calculate $\mathbb{E}\Big[(X + Y)^2\Big]$

   (d) (2 points) Calculate the standard deviation of $X + Y$.

9. (a) (4 points) In the months June, July, and August, the number of murders in a city occurring in that month is distributed as a Poisson random variable with mean 1.

   In the other 9 months of the year, the number of murders is modeled by a Poisson random variable with mean 0.5.

   Assume that these 12 random variables are independent.

   Calculate the probability that exactly two accidents occur in July through November.

10. The number of phone calls to a service center, N, is Poisson distributed with mean λ. The parameter λ is a random variable that is determined by the level of activity on that day, and is uniformly distributed on the interval $[0, 100]$.

    (a) (3 points) Calculate $\mathbb{E}(N)$.

    (b) (3 points) Calculate $\mathrm{Var}(N)$.