The Neyman-Pearson Lemma

April 22, 2020

The Neyman-Pearson Lemma will give us the best test when we assume a certain type of hypothesis test. We will give a definition that specifies certain a type of hypothesis. Then we give a way to specify that a test is the best. Then we will be able to state the lemma.

1 Neyman-Pearson Lemma

Definition 1.1. *A hypothesis is called a **simple hypothesis** if it uniquely specifies the distribution of the sample random variables. A hypothesis that is not simple is called **composite**.*

In this class, we often chose the null hypothesis to a be a simple hypothesis. Indeed, if we specify everything about the random sample except its mean, θ, and then choose $H_0 : \theta = \theta_0$, for a specified value of θ_0, then when H_0 is assumed the entire distribution of the sample random variables is now known.

On the other hand, we usually choose the alternative hypothesis to be a composite hypothesis. Indeed, if $H_a : \theta < \theta_0$, then if you're told that H_a is true, you still don't know what θ is, only that it is some number less that θ_0, so we don't actually know the pdf of the sample random variables.

One important thing to note, is that our null hypothesis is not always a simple hypothesis. For example, if the random sample comes from a normal distribution with unknown μ and σ^2, and we choose $H_0 : \theta = \theta_0$, then this is not a simple hypothesis because it did not specify σ^2 and therefore did not specify everything about the pdf of the random sample.

When designing a hypothesis test for the simple hypothesis $H_0 : \theta = \theta_0$ against the simple hypothesis $H_a : \theta = \theta_a$, we start by choosing an acceptable α, this is simply a matter of how confident you want to be. Recall $\alpha = \text{power}(\theta_0)$ by their definitions. We then want to chose the hypothesis with that largest power at θ_a. Designing the hypothesis test will amount to choosing a test statistic, then the distribution of the test statistic gives a RR for the given α. We give the test with the largest power a name:

Definition 1.2. *When testing the simple hypothesis $H_0 : \theta = \theta_0$ against the simple hypothesis $H_a : \theta = \theta_a$, the test with the largest power(θ_a) is called the **most powerful test**.*

The Neyman-Pearson Lemma gives us a way to find the most powerful test when H_0 and H_a are both simple hypothesises.

From the pervious discussion, you should question the usefulness of a test that requires each hypothesis to be simple, as our H_a is rarely simple, and sometimes our H_0 is not simple. After stating the lemma and an example, we will discuss how the lemma can still be used when H_a is not simple. The case when H_0 is not simple will be handled in the next section.

LEMMA 1.3 (The Neyman-Pearson Lemma). *Let $Y_1,...,Y_n$ be a random sample with likelihood function $L(\theta)$. The most powerful test of the simple hypothesis $H_0:\theta=\theta_0$ against the simple hypothesis $H_a:\theta=\theta_a$ at level α has a rejection region of the form:*

$$\frac{L(\theta_0)}{L(\theta_a)}<k$$

where k is chosen so that the probability of a type I error is α.

The main statement should be someone believable, but how to apply the lemma might not. The lemma says that given you data, compute the likelihood function at θ_0 and θ_a. Intuitively, you would like to say the θ with the larger likelihood function is the correct one. This would correspond to $k=1$ in the above lemma. But we don't simply want to choose $k=1$ because we know there is some chance that randomly the wrong θ looks for likely. So instead of choosing $k=1$ we choose k such that under H_0 probability of $\frac{L(\theta_0)}{L(\theta_a)}<k$ is α.

Let's look at an example:

EXAMPLE 1.4. *Let $Y_1,...,Y_4$ be a random sample with pdf:*

$$f_{Y|\theta}(y|\theta)=\frac{1}{2\theta^3}y^2e^{-y/\theta}$$

for $y>0$, and 0 otherwise, where $\theta>0$ is an unknown parameter.

Find the Rejection Region for the most powerful test for $H_0:\theta=3$ against $H_a:\theta=6$.

Solution:

First note that the Y_i are gamma random variables with $\alpha=3$ and $\beta=\theta$.

Since the Null Hypothesis and the Alternative Hypothesis are both simple we can apply the Neyman-Pearson Lemma.

We begin by computing the likelihood function, which we then evaluated $\theta=3$ and 6. We multiply the pdf's of Y_1 through Y_4

$$L(y_1,y_2,y_3,y_4|\theta)=\prod_{i=1}^{4}\frac{1}{2\theta^3}y_i^2e^{-y_i/\theta}=\frac{1}{2^4\theta^{12}}\ y_1^2\ y_2^2\ y_3^2\ y_4^2\ e^{\frac{-y_1-y_2-y_3-y_4}{\theta}}$$

Then by the Neyman-Pearson Lemma we consider the ratio:

$$\frac{L(3)}{L(6)}=\frac{\frac{1}{2^43^{12}}\ y_1^2\ y_2^2\ y_3^2\ y_4^2\ e^{\frac{-y_1-y_2-y_3-y_4}{3}}}{\frac{1}{2^46^{12}}\ y_1^2\ y_2^2\ y_3^2\ y_4^2\ e^{\frac{-y_1-y_2-y_3-y_4}{6}}}$$

where we used that we are given $\theta_0=3$ and $\theta_a=6$.

Then we simplify:

$$\frac{L(3)}{L(6)}=\frac{6^{12}}{3^{12}}e^{-(y_1+y_2+y_3+y_4)\left(\frac{1}{3}-\frac{1}{6}\right)}$$

We will now apply the Neyman-Pearson Lemma to the above expression. This requires studying its distribution under H_0. We'll need to use that, under H_0, $Y_1+Y_2+Y_3+Y_4$ is a gamma random variable with $\alpha=3*4=12$ and $\beta=\theta_0=3$. So now we solve for $Y_1+Y_2+Y_3+Y_4$, in

$$\frac{6^{12}}{3^{12}}e^{-(y_1+y_2+y_3+y_4)\left(\frac{1}{3}-\frac{1}{6}\right)}<k$$

We multiply both sides by $\frac{3^{12}}{6^{12}}$

$$e^{-(y_1+y_2+y_3+y_4)\left(\frac{1}{3}-\frac{1}{6}\right)} < \left(k\frac{3^{12}}{6^{12}}\right).$$

Then we take the ln of both sides.

$$-(y_1+y_2+y_3+y_4)\left(\frac{1}{3}-\frac{1}{6}\right) < \ln\left(k\frac{3^{12}}{6^{12}}\right)$$

Now we multiply by $-\left(\frac{1}{3}-\frac{1}{6}\right)^{-1} = -6$, because this number is negative, we switch the direction of the inequality.

$$(y_1+y_2+y_3+y_4) > -6*\ln\left(k\frac{3^{12}}{6^{12}}\right). \tag{1.1}$$

Then we use:

$$\mathbb{P}(Y_1+Y_2+Y_3+Y_4 > 54.62254) = .05$$

Where I compute 54.62254 using the R command qgamma(.95,shape=12,scale = 3).

So our test statistic is $Y_1+Y_2+Y_3+Y_4$ and RR is

$$Y_1+Y_2+Y_3+Y_4 > 54.62254.$$

This is the answer.

Some remarks: We don't actually care about the exact value of k. The only things we use about (1.1) is that $Y_1+Y_2+Y_3+Y_4$ is a usable test statistic and the direction of the inequality.

We used the value $\theta_0 = 3$ to compute the rejection region, it gave us the distribution of $Y_1+Y_2+Y_3+Y_4$ under H_0. We didn't really use the exact value of $\theta_a = 6$, in fact the only place it comes up is when we multiplied both sides by $-\left(\frac{1}{3}-\frac{1}{6}\right)^{-1}$. Because $6 > 3$ this number is negative. In fact for any $\theta_a > 3$ we would have the exact same same rejection region. On the other hand if our alternate hypothesis was with a $\theta_a < 3 = \theta_0$ then this term would be positive and we wouldn't flip the inequality.

The final remark of the last example (that we didn't use very strongly the value of θ_a) shows us the the Neyman-Pearson lemma can be useful for some composite tests and motives the next definition:

DEFINITION 1.5. *When testing the simple hypothesis $H_0 : \theta = \theta_0$ against the composite alternative hypothesis H_a, if a test has the largest power(θ) for all θ in H_a then it is called the **uniformly most powerful test**.*

In the last example, the remark at the end shows that the hypothesis test with RR

$$Y_1+Y_2+Y_3+Y_4 > 54.62254$$

is the uniformly most powerful test of $H_0 : \theta = 3$ against $H_a : \theta > 3$ at the $\alpha = .05$ level.

Note that a uniformly most powerful test might not exist. In fact we should only expect it for one-sided alternative hypothesis tests.

In general, to find the uniformly most powerful test (when it exist), you begin by picking an arbitrary θ_a in H_a. For this particular choice you apply the Neyman-Pearson lemma to compute the RR. If you only use properties about θ_a that hold for all θ_a in the entire H_a, then you have found the uniformly most test. If this is impossible, then there is no uniformly most powerful test.

Let's do another example of finding the uniformly most powerful test.

EXAMPLE 1.6. *Let $X_1,X_2,...,X_n$ be a random sample of Poisson random variables with unknown mean $\lambda > 0$. Find the Rejection Region for the uniformly most powerful test of $H_0: \lambda = \lambda_0$ against the alternative hypothesis $H_a: \lambda < \lambda_0$ at level α. Where λ_0 is some fixed number.*

Solution: Before we start recall that $X_1+X_2+...+X_n$ is also a Poisson random variable, but with mean $n\lambda$, and that the pmf of X_1 is

$$p_{X_i}(x) = e^{-\lambda}\frac{\lambda^x}{x!}$$

for all non-negative integers x.

From the Neyman-Pearson Lemma we compute Likelihood function

$$L(\lambda) = \prod_{i=1}^{n} e^{-\lambda}\frac{\lambda^{x_i}}{x_i!} = e^{-n\lambda}\frac{\lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}$$

then we take the ratio between $L(\lambda_0)$ and $L(\lambda_a)$ for some $\lambda_a < \lambda_0$:

$$\frac{L(\lambda_0)}{L(\lambda_a)} = \frac{e^{-n\lambda_0}\frac{\lambda_0^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}}{e^{-n\lambda_a}\frac{\lambda_a^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}} = \frac{e^{-n\lambda_0}\lambda_0^{\sum_{i=1}^n x_i}}{e^{-n\lambda_a}\lambda_a^{\sum_{i=1}^n x_i}} = e^{-n(\lambda_0-\lambda_a)}\left(\frac{\lambda_0}{\lambda_a}\right)^{\sum_{i=1}^n x_i}$$

Then the RR will be when this quantity is less than some constant k.

$$e^{-n(\lambda_0-\lambda_a)}\left(\frac{\lambda_0}{\lambda_a}\right)^{\sum_{i=1}^n x_i} < k$$

We now solve for $\sum_{i=1}^n x_i$, because we know the distribution of the random variable $\sum_{i=1}^n X_i$, so we can use it to determine RR.

Multiplying both sides by (the positive number) $e^{n(\lambda_0-\lambda_a)}$ and then taking the ln gives:

$$\left(\sum_{i=1}^n x_i\right)\ln\left(\frac{\lambda_0}{\lambda_a}\right) < \ln(e^{n(\lambda_0-\lambda_a)}k)$$

Then we multiply by $\ln\left(\frac{\lambda_0}{\lambda_a}\right)^{-1}$, because $\lambda_a < \lambda_0$ this number is positive so we have

$$\left(\sum_{i=1}^n x_i\right) < \ln(e^{n(\lambda_0-\lambda_a)}k)\ln\left(\frac{\lambda_0}{\lambda_a}\right)^{-1} = k'$$

Where we now call the term on the right k', because we just care about the direction of the inequality.

So RR is

$$\left(\sum_{i=1}^n x_i\right) < k'$$

Where k' is such that

$$\mathbb{P}(X \leq k') = \alpha$$

where X is a Poisson random variable with mean $n\lambda_0$ (note we're using H_0 to determine RR, as usual).

Since we only used properties of θ_a that hold for every value in H_a (namely $\theta_a \leq \theta_0$) this is the uniformly most powerful test.

Recall when doing estimation, we usually either considered examples like that last one, where we estimated the mean, or we did examples where the support of the pdf was unknown. Let's look at an example of the second type in this context:

EXAMPLE 1.7. *Let $Y_1,Y_2,...,Y_n$ be a random sample from a uniform distribution over the interval $(0,\theta)$. Find the most powerful α-level test for testing $H_0:\theta=\theta_0$ against $H_a:\theta=\theta_a$, where $\theta_a<\theta_0$. Is this the uniformly most powerful test for $H_0:\theta=\theta_0$ against $H_a:\theta<\theta_0$?*

Solution:

Recall the likelihood function is

$$L(\theta)=\frac{1}{\theta^n} \text{ if } \theta>y_{(n)}$$

and 0 otherwise. Where $y_{(n)}=\max\{y_1,y_2,...,y_n\}$

We first note $y_{(n)}$ cannot be bigger than θ_0 (or θ_0 cannot be smaller than $y_{(n)}$, depending on your perspective). If $\theta_a<y_{(n)}<\theta_0$, then we do not reject H_0, because it would be impossible for θ_a to be upper edge of the support of the pdf. So the interesting region to consider is $y_{(n)}<\theta_a$, where we have:

$$\frac{L(\theta_0)}{L(\theta_a)}=\frac{\frac{1}{\theta_0^n}}{\frac{1}{\theta_a^n}}=\frac{\theta_a^n}{\theta_0^n}$$

For $y_{(n)}<\theta_a$. Note that here we used $\theta_a<\theta_0$.

We set this less than some constant k and get

$$\frac{\theta_a^n}{\theta_0^n}\mathbb{I}_{y_{(n)}<\theta_a}<k$$

Where $\mathbb{I}_{y_{(n)}<\theta_a}=1$ if $y_{(n)}<\theta_a$ and zero otherwise.

The function $\frac{\theta_a^n}{\theta_0^n}\mathbb{I}_{y_{(n)}<\theta_a}$ only takes on two values, if $y_{(n)}$ is small it is zero and then as $y_{(n)}$ increases, at some point it jumps up to $\frac{\theta_a^n}{\theta_0^n}$.

The above inequality tells us to then reject θ_0 when $Y_{(n)}$ is small.

We've seen that the pdf of $Y_{(n)}$ is

$$f_{Y_{(n)}}(u)=n\frac{y^{n-1}}{\theta^n}\mathbb{I}_{y<\theta}$$

So under H_0 we replace θ with θ_0 and then we want a such that

$$\mathbb{P}(Y_{(n)}<a)=\alpha$$

or

$$\int_0^a n\frac{y^{n-1}}{\theta_0^n}dy = \alpha$$

Computing the integral gives:

$$\frac{a^n}{\theta_0^n} = \alpha$$

Solving for a:

$$a = (\alpha\theta_0^n)^{1/n}$$

So the rejection region is

$$Y_{(n)} < \alpha^{1/n}\theta_0$$

Since RR and its derivation never used the actual value of θ_a, only that it is less than θ_0 this is the uniformly most powerful test for $H_0: \theta = \theta_0$ against $H_a: \theta < \theta_0$.