Comment on functions of random variables

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These notes are just a remark showing the difference between computing $\mathbb{E}[X^k]$ and $f_{X^k}(x)$. If you have any confusion about these two ideas, hopefully these notes can clarify it.

Earlier in this class we were interested in computing moments of a random variable X. Which for any positive integer, k, are given by $\mathbb{E}[X^k]$. If X is a continuous random variable, this is computed by

$$\mathbb{E}[X^k] = \int x^k f_X(x)\ dx$$

where $f_X(x)$ is the pdf of X.

Now we have learned to compute not just the expectation of X^k but also the pdf of X^k. We don't exactly have a formula for computing this, but our general method is

$$\begin{aligned} f_{X^k}(x) &= \frac{d}{dx} F_{X^k}(x) \\ &= \frac{d}{dx} \mathbb{P}(X^k \leq x) \\ &= \frac{d}{dx} \int_{t^k \leq x} f_X(t)\ dt \end{aligned}$$

then the set $t^k \leq x$ is different depending on if k is even or odd. When k is odd $t^k \leq x$ means $t \leq x^{1/k}$ and if k is even then $t^k \leq x$ means $-x^{1/k} \leq t \leq x^{1/k}$.

So then when k is odd

$$\begin{aligned} f_{X^k}(x) &= \frac{d}{dx} \int_{t^k \leq x} f_X(t) dt \\ &= \frac{d}{dx} \int_{-\infty}^{x^{1/k}} f_X(t) dt \\ &= \frac{d}{dx} \left(F_X(x^{1/k}) \right) \\ &= f_X(x^{1/k}) \frac{1}{k x^{(k-1)/k}} \end{aligned}$$

(Don't forget the chain rule at the end) and when k is even

$$\begin{aligned}
f_{X^k}(x) &= \frac{d}{dx}\int_{t^k \le x} f_X(t)dt \\
&= \frac{d}{dx}\int_{-x^{1/k}}^{x^{1/k}} f_X(t)dt \\
&= \frac{d}{dx}\left(F_X(x^{1/k}) - F_X(-x^{1/k})\right) \\
&= f_X(x^{1/k})\frac{1}{kx^{(k-1)/k}} + f_X(-x^{1/k})\frac{1}{kx^{(k-1)/k}}
\end{aligned}$$

Once we have the pdf of X^k we have another way to compute $\mathbb{E}[X^k]$:

$$\mathbb{E}[X^k] = \int x \ f_{X^k}(x) \ dx.$$

This is generally not a better way to compute $\mathbb{E}[X^k]$, but is nevertheless true. You can check that these two formulas are the same by doing the u-substitution, $u^k = x$.

The usefulness of knowing the pdf X^k, is we can now determine the probability X^k takes a value in the interval $[a,b]$ by computing

$$\mathbb{P}(a \le X^k \le b) = \int_a^b f_{X^k}(x) \ dx.$$