Conditional Expectation

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If Y is a random variable. When we compute something that conditions on Y, we are assuming we know the value Y takes, and seeing how that effects other random variables. This is similar to in Chapter 2 when we conditioned on one event, that is assumed the event held, and saw how that effects the other random variables. Sometimes we conditioned on events because we are told they happen and sometimes we used it as a tool to make other calculations easier. We will do the same here.

With conditional expectations, sometimes we explicitly write out the the value Y takes, something like $\mathbb{E}[X|Y=y]$, where y is a number, and sometimes we don't, something like $\mathbb{E}[X|Y]$. In the second case, we think of $\mathbb{E}[X|Y]$ as still being a random variable, but when we're computing $\mathbb{E}[X|Y]$, we think of it as fixed.

The conditional expectation $\mathbb{E}[X|Y]$ can be computed in several ways depending on the information you are given.

If you start with the joint distribution of X and Y, $f_{X,Y}(x,y)$, then it is often best to compute the conditional pdf of X by:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_{X,Y}(x,y)}{\int f_{X,Y}(x,y)dx}.$$

Then the conditional expectation can be computed, similar to a normal expectation, but with the conditional pdf instead:

$$\mathbb{E}[X|Y] = \int x f_{X|Y}(x|Y)dx$$

notice that the x is integrated out, but the answer will still involve Y.

Note that if you were actually interested in $\mathbb{E}[X]$ and started with the joint distribution, there is no need to use the conditional expectation. You should just do

$$\mathbb{E}[X] = \iint x f_{X,Y}(x,y)dxdy = \int x f_X(x)dx.$$

If you start with a conditional distribution, can skip to $\mathbb{E}[X|Y] = \int x f_{X|Y}(x|Y)dx$. Sometimes it is even easier, if we are working with random variables we know well. For example, if X is a exponential random variable with mean Y, then $\mathbb{E}[X|Y] = Y$, no matter what the distribution of Y is. The same holds for the conditional variance, in this case $\mathrm{Var}(X|Y) = Y^2$, where I just used the formula for the variance of an exponential random variable.

Often times, when we are given the conditional distribution we will actually be interested in the (unconditional) expectation. In this case the law of iterated expectation is useful:

$$\mathbb{E}[X] = \mathbb{E}\Big[\mathbb{E}[X|Y]\Big]$$

So if X is a exponential random variable with mean Y then $\mathbb{E}[X] = \mathbb{E}[Y]$, and then the problem should give some information about Y, that let's you finish the problem.

Most of the times for these types of problems the distribution of X can be complicated and you don't want to write down any pdf/pm's and try to integrate/ sum them.