# Review of Multivariate Random Variate

April 20, 2020

We introduced a lot of new ideas and questions in this Chapter, so we'll do some examples where we try to use all the ideas. We'll see what some typical problems in both the discrete and continuous case.

## 1 Discrete Random Variables

There are 3 basic types of problems we consider. One is where you're directly given the joint pmf of the random variables. The second is where the marginal distribution of one random variable is given and the conditional distribution of the other given the is given. In the third, the random variables are independent and their marginal distributions are given. In this case, the joint distribution is just the product of the marginal distributions, and many of the things we look at in this section are not so interesting, so we'll focus on the first two cases here.

Example 1.1. *The joint distribution of the amount of rainfall in two cities (rounded down to the nearest inch) is given by following table:*

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | .2 | .05 | 0 | 0 |
| 1 | .1 | .2 | .1 | 0 |
| 2 | 0 | .05 | .1 | 0 |
| 3 | 0 | 0 | .15 | .05 |

*The number of inches in city A is given by the columns and the number of inches in city B is given by the rows. So for example, the probability it rains 2 inches in city A and 3 inches in city B is .15.*

*(a) Show that this is a joint pmf.*

*(b) What is the marginal distribution of rainfall in city A?*

*(c) Given that it rained 1 inch in city B, what is the distribution of rain in city A?*

*(d) What is the expectation of the rainfall in city A?*

*(e) What is the conditional expectation of rainfall in city A, given that it rained one inch in city B?*

*(f) What is the covariance between the amount of rainfall in city A and city B?*

**Solution:** Let $A$ be the amount of rainfall in city A and let $B$ be the amount of rainfall in city B.

(a) Since all the numbers above are clearly positive, the fact that this is a joint pmf follows by noting the sum of all the numbers in the table are 1.

(b) To compute the marginal distribution of $A$, we sum over all possibilities of rain in city B:

$$p_A(0) = .2 + .1 = .3$$

$$p_A(1) = .05 + .2 + .05 = .3$$

$$p_A(2) = .1 + .1 + .15 = .35$$

$$p_A(3) = .05$$

(c) We begin by computing the probability $B$ equals 1, $p_B(1) = .1 + .2 + .1 = .4$. To compute this marginal pmf

$$p_{A|B}(0|1) = \frac{p_{A,B}(0,1)}{p_B(1)} = \frac{.1}{.4} = .25$$

$$p_{A|B}(1|1) = \frac{p_{A,B}(1,1)}{p_B(1)} = \frac{.2}{.4} = .5$$

$$p_{A|B}(2|1) = \frac{p_{A,B}(2,1)}{p_B(1)} = \frac{.1}{.4} = .25$$

$$p_{A|B}(3|1) = \frac{p_{A,B}(3,1)}{p_B(1)} = \frac{0}{.4} = 0$$

(d) We could compute this directly from the jpmf, by summing over 16 terms, but since we computed the pmf of $A$ above we can use that:

$$\mathbb{E}[A] = 0 * .3 + 1 * .3 + 2 * .35 + 3 * .05 = 1.15$$

(e) We compute this conditional expectation by using the conditional pmf we computed above:

$$\mathbb{E}[A|B = 1] = 0 * .25 + 1 * .5 + 2 * .25 + 3 * 0 = 1$$

(f) To compute the covariance, we need $\mathbb{E}[AB]$, $\mathbb{E}[A]$ and $\mathbb{E}[B]$. The $\mathbb{E}[A]$ was computed above. The other two we'll compute directly from the joint pmf.

$$\begin{aligned}
\mathbb{E}[AB] &= \sum_{a=0}^{3} \sum_{b=0}^{3} a \ b \ p_{A,B}(a,b) \\
&= 0*0*.2 + 0*1*.05 + 0*2*0 + 0*3*0 \\
&\quad + 1*0*.1 + 1*1*.2 + 1*2*.1 + 1*3*0 \\
&\quad + 2*0*0 + 2*1*.05 + 2*2*.1 + 2*3*0 \\
&\quad + 3*0*0 + 3*1*0 + 3*2*.15 + 3*3*.05 \\
&= 2.25
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}[B] &= \sum_{a=0}^{3}\sum_{b=0}^{3} b\ p_{A,B}(a,b) \\
&= 0*.2+0*.05+0*0+0*0 \\
&\quad +1*.1+1*.2+1*.1+1*0 \\
&\quad +2*0+2*.05+2*.1+2*0 \\
&\quad +3*0+3*0+3*.15+3*.05 \\
&= 0*(.2+.05+0+0) \\
&\quad +1*(.1+.2+.1+0) \\
&\quad +2*(0+.05+.1+0) \\
&\quad +3*(0+0+.15+.05) \\
&= 1.3
\end{aligned}
$$

Putting this together we have:

$$\mathrm{Cov}(A,B) = \mathbb{E}[AB] - \mathbb{E}[A]\mathbb{E}[B] = 2.25 - 1.15*1.3 = .755$$

Now we consider an example where a conditional distribution is given.

EXAMPLE 1.2. *We have a biased coin (probability of heads is equal to* $1/4$*). Consider the following 2 step process:*

*In the first step we flip the coin until we get a heads. Let* $X$ *denote the number of trials before the first heads occurs (note this is not quite a geometric as we defined in class, but 1 minus that. So for example, if you flipped 4 tails and then a head,* $X$ *would be 4.).*

*In the second step we flip the coin* $X$ *more times. Let* $Y$ *be the number of heads in the second step. (Note that if we flipped a head on the first trial* $X$ *would be* $0$*, and so would* $Y$*.)*

(a) *What is the joint pmf of* $X$ *and* $Y$*?*

(b) *What is the marginal pmf of* $Y$*?*

(c) *What is the conditional pmf of* $X$*, given* $Y$*?*

**Solution:**

(a) The problem gives us (in words) the marginal pmf of $X$ and the conditional pmf of $Y$. We'll write these out mathematically, then multiplying them gives the joint pmf.

The marginal pmf of $X$ is:

$$p_X(n) = (3/4)^n(1/4)$$

for $n = 0,1,2,\ldots.$ This is similar to the geometric, but the exponent on the $3/4$ is $n$ instead of $n-1$ because we're just counting the number of tails.

$$p_{Y|X}(k|n) = \binom{n}{k}(1/4)^k(3/4)^{n-k}$$

For $k=0,1,2,...,n$. The conditional distribution is just binomial.

$$p_{Y,X}(k,n)=p_{Y|X}(k|n)p_X(k)=\binom{n}{k}(1/4)^k(3/4)^{n-k}(3/4)^n(1/4)=\binom{n}{k}(1/4)^{k+1}(3/4)^{2n-k}$$

For $n=0,1,2,...$ and $k=0,1,2,...,n$.

(b) To get the marginal distribution we need to sum of $X$. Be careful with the bounds of the sum.

$$p_Y(k)=\sum_n p_{Y,X}(k,n)=\sum_{n=k}^{\infty}\binom{n}{k}(1/4)^{k+1}(3/4)^{2n-k}=\frac{4}{7}\left(\frac{3}{7}\right)^k$$

for $k=0,1,2,...$ I computed the infinite sum on my computer, but it can be done (after a bit of rearrangement) by computing the $k^{th}$ derivative of a geometric series. You can check that this is a pmf, because the sum over all the terms is 1.

(c)

$$p_{X|Y}(n|k)=\frac{p_{Y,X}(k,n)}{p_Y(k)}=\frac{\binom{n}{k}(1/4)^{k+1}(3/4)^{2n-k}}{\frac{4}{7}\left(\frac{3}{7}\right)^k}$$

for $n=k,k+1,...$. This could be simplified a bit, but won't look much nicer.

## 2 Continuous Random Variables

EXAMPLE 2.1. *The amount of time, $X$, until the second person arrives at a store is a random variable with pdf:*

$$f_X(x)=\frac{x}{100}e^{-x/10}$$

*for all $x>0$.*

*The time, $Y$, that the first person arrives is uniformly distributed between 0 and $X$.*

*(a) What is the joint distribution of $X$ and $Y$?*

*(b) What is the marginal distribution of $Y$?*

*(c) What is the conditional distribution of $X$ given $Y$?*

*(d) What is the covariance between $X$ and $Y$?*

*(e) What is the conditional expectation of $Y$ given $X$?*

*(f) What is the conditional expectation of $X$ given $Y$?*

**Solution:** The conditional pdf of $Y$ is $f_{Y|X}(y|x)=1/x$ if $0\le y\le x$ and 0 otherwise.

(a) To compute the joint distribution, we multiply the conditional distribution by the marginal distribution of what we condition on.

$$f_{X,Y} = f_{Y|X}(y|x) f_X(x) = \frac{1}{x}\frac{x}{100}e^{-x/10} = \frac{1}{100}e^{-x/10}$$

for $0 \le y \le x$, and 0 otherwise.

(b) To compute the marginal distribution of $X$, we integrate out $Y$.

$$f_Y(y) = \int f_{X,Y}\, dx = \int_y^{\infty} \frac{1}{100}e^{-x/10}dx = \frac{1}{10}e^{-y/10}$$

for $y > 0$.

(c) What is the conditional distribution of $X$ given $Y$?

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_y(y)} = \frac{\frac{1}{100}e^{-x/10}}{\frac{1}{10}e^{-y/10}} = \frac{1}{10}e^{-(x-y)/10}$$

for $x \ge y$.

(d) To compute $\text{Cov}(X,Y)$ we need $\mathbb{E}[XY]$, $\mathbb{E}[X]$, and $\mathbb{E}[Y]$.

$$\begin{aligned}
\mathbb{E}[XY] &= \int xy\ f_{X,Y}(x,y)\ dxdy \\
&= \int_0^{\infty}\int_0^{x} xy\frac{1}{100}e^{-x/10}dydx \\
&= \int_0^{\infty} x\frac{1}{100}e^{-x/10}\frac{y^2}{2}\Big|_0^x dx \\
&= \int_0^{\infty} \frac{x^3}{2}\frac{1}{100}e^{-x/10}dx \\
&= 300
\end{aligned}$$

Since we have computed the marginal pdfs, we'll use them to compute the expectations:

$$\mathbb{E}[X] = \int x f_X(x) dx = \int_0^{\infty} x\ \frac{x}{100}e^{-x/10}\ dx = 20$$

$$\mathbb{E}[Y] = \int y f_Y(Y) dy = \int_0^{\infty} y\ \frac{1}{10}e^{-y/10}\ dy = 10$$

So $\text{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 300 - 200 = 100$.

(e)

$$\mathbb{E}[Y|X] = \int y f_{Y|X}(y|X) dy = \int_0^{X} y\frac{1}{X}dy = \frac{X}{2}$$

(f)

$$\mathbb{E}[X|Y] = \int x f_{X|Y}(x|Y) dx = \int_Y^{\infty} x\frac{1}{10}e^{-(x-Y)/10}dx = Y + 10$$

EXAMPLE 2.2. *Let $X$ and $Y$ have joint distribution*

$$f_{X,Y}(x,y) = x+y$$

*for $0 \leq x \leq 1$ and $0 \leq y \leq 1$.*

(a) *Compute the marginal distribution of $X$.*

(b) *Compute the conditional distribution of $Y$, given $X = x$.*

(c) *Compute the covariance between $X$ and $Y$.*

(d) *Compute the conditional expectation of $Y$, given $X$.*

**Solution:**

(a)

$$f_X(x) = \int f_{X,Y}(x,y)\ dy = \int_0^1 (x+y)\ dy = xy + \frac{y^2}{2}\Big|_0^1 = x + \frac{1}{2}$$

(b)

$$f_{Y|X}(y|x) = \frac{f_{Y,X}(y,x)}{f_X(x)} = \frac{x+y}{x+\frac{1}{2}}$$

for $y$ between 0 and 1, and 0 otherwise.

In this formula, you should think of $x$ as a fixed number, and then $f_{Y|X}(y|x)$ is a function of $y$. We then can integrate it as a function of $y$, to compute the probability $Y$ takes on certain value, given that $X = x$.

(c) To compute the covariance, $\mathbb{E}[XY]$, $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

$$\mathbb{E}[XY] = \iint xy(x+y)\ dx\ dy = 1/3$$

$$\mathbb{E}[X] = \iint x(x+y)\ dx\ dy = 7/12$$

$$\mathbb{E}[Y] = \iint y(x+y)\ dx\ dy = 7/12$$

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \frac{1}{3} - \frac{7}{12}\frac{7}{12} = -1/144$$

(d)

$$\mathbb{E}[Y|X] = \int y f_{Y|X}(y,X) = \int_0^1 y\frac{X+y}{X+\frac{1}{2}} dy = \frac{2+3X}{3+6X}$$

# 3 Mixing Continuous and Discrete Variables

The book doesn't formally introduce it, but the idea of continuing a one type of random variable on the other works quite similarly. Here is an example:

EXAMPLE 3.1. *Let $X$ be a uniformly distribution random variable between 0 and 1. Let $Y$ be a geometric random variable with parameter $X$.*

*(a) Compute the conditional expectation of $Y$, given $X$.*

*(b) Compute the expectation of $Y$.*

**Solution:**

(a) The conditional pmf of $Y$ is

$$p_{Y|X}(y|x) = x(1-x)^{y-1}$$

for $y$ a positive integer. So the conditional expectation of $Y$ given $X$ is

$$\mathbb{E}[Y|X] = \sum_{y=1}^{\infty} y(1-X)^{y-1}X = \frac{1}{X}$$

where we have computed the infinite sum like we did for the expectation of a geometric random variable.

(b) Using the law of iterated expectation:

$$\mathbb{E}[Y] = \mathbb{E}\Big[\mathbb{E}[Y|X]\Big] = \mathbb{E}[\frac{1}{X}] = \int 1/x f_X(x)dx = \int_0^1 1/x dx = \infty$$

So the unconditional expectation is actually infinite.