**Summary of Critical Values, Derivative Tests and Concavity**

If *f '*(*c*) = 0, or if *f*'(*c*) does not exist (DNE), then *c* is a critical number. The function could have a local extreme at *c* (including a nondifferentiable cusp or corner) or it could have an *inflection point.*

*Before* invoking the first derivative test to see if the derivative changes sign, we *could* go straight to the *second derivative test*, testing *f''*(*c*) and drawing conclusions as follows:

* If *f''*(*c*) > 0, the function is concave up at *c* because the trend of the slope of the tangent is to increase. *f*(*c*) is a local min.
* If *f''*(*c*) < 0, the function is concave down at *c* because the trend of the slope of the tangent is to decrease. *f*(*c*) is a local max.
* If *f''*(*c*)= 0, we could have a local max or a local min, **or** we could have an inflection point. How do we decide what it is? There are a couple of things we can do:

EITHER

1. Resort to the f*irst derivative test*, checking values on either side of *c* to see if *f'*(*c*) changes sign.

If it does, we have a local extreme at *x* = *c*.

If it doesn't, we have an inflection point at *x* = *c*.

OR

2. Stay with the *second derivative test*, testing for a value on either side of *c* to see if *f''*(*x*) changes sign.

If *f''*(*x*) < 0 to the left of *c* and > 0 to the right of *c* (or if it is > 0 to the left and < 0 to the right), then *c* is clearly an inflection point because concavity has changed.

If *f''*(*x*) is > 0 on *both* *sides* then *x = c* is a local min

If *f''*(*x*) < 0 on *both* sides, then *x* = *c* is a local max.

Following are two examples that are illustrative because the functions are so simple you can already sketch them. Observe the derivative tests vis-a-vis the sketches to lock in your understanding.

**Example 1: Sketch the function *f*(*x*) = *x*4**

*f'*(*x*) = 4*x*3 = 0 at *x* = 0; *f''*(*x*) = 12*x*2 = 0 at *x* = 0, also. What kind of critical point, then, is *c* = 0 ? There are two ways to find out:

*First derivative test*: Checking values into f'(x) on either side of 0, *f'*(-1) = 4(-1)3 = -4 and *f'*(1) = 4(1)3 = 4. Because *f'* changes sign, negative to positive, *c* = 0 is a local min.

*Second derivative test:* Checking values of *f''*(*x*) on either side of 0, *f''*(-1) = 12(-1)2 = 12 and *f''*(1) = 12(1)2 = 12. No change in sign, *f''* is positive on either side, so the function is concave up and *c* = 0 is a local min.

**Example 2: Sketch the function *f'*(*x*) = *x*3**

*f'*(*x*) = 3*x*2 = 0 at *x* = 0; *f''*(*x*) = 6*x* = 0 at *x* = 0 also. So, what kind of critical point is *c* = 0 ? There are two ways to find out:

*First derivative test:* Checking values of *f'*(*x*) on either side of 0, *f'*(-1) = 3(-1)2 = 3, and *f'*(1) = 3(1)2 = 3. Thus, the function is increasing on either side of *c* = 0, so *c* is an inflection point.

*Second derivative test:* Checking values of *f''*(*x*) on either side of 0, *f''*(-1) = 6(-1) = -6 < 0. *f''*(1) = 6(1) = 6 > 0. The change in sign indicates the graph is concave down to the left of *x* = 0 and concave up to the right of it, at *x* = 0 is an inflection.

**Example 3: Sketch the function *f'*(*x*) = *x*1/3**

*f'*(*x*) = 1/3*x*2/3. The function is differentiable on its domain, but its derivative cannot equal zero. However, *f'*(0) does not exist (division by zero). In the DNE sense, *c =* 0 is critical value of the function. (You should see from your graph that the tangent line to the function at *x* = 0 is a vertical line.)

*First derivative test:* The function is increasing everywhere, as is easily seen in the graph; algebraically, *f'*(*x*) is positive on the domain, observable by the fact that *x*2/3 is the square of a cube root. Thus, by the first derivative test, the function is everywhere increasing. There is no max or min. Is it an inflection point ?

*f''*(*x*) = -2/9*x*5/3

*Second derivative test: f''*(*x*) DNE at x = 0, for the same reason (division by zero). Checking values of *f''*(*x*) on either side of 0, *f''*(-1) = (-2/9)(-1)5/3= 2/9 > 0 (concave up). *f'*(1) = (-2/9)(1)5/3 = -2/9 < 0 (concave down). *x* = 0 is a point of inflection.