Math 220 - Calculus f. Business and Management - Worksheets 5 & 6

Solutions for Worksheet 5 & 6 - Exponents and Logs

Solving exponential equations

Exercise 1:

Find the value of x in each expression.

$$\text{1a: } 5^{2x+3} = 5^4, \quad \text{1b: } 3^{4x-7} = 3^{6-2x}, \quad \text{1c: } 7^{x^2-8} = 7^{2x},$$
$$\text{1d: } 4^{x^3} = 4^x, \quad \text{1e: } 2^x = 4^{x+5}, \quad \text{1f: } 9^{-4x} - 3^{2x-3} = 0$$

Solution to #1:

1a: Since the bases are equal, exponents must be equal. So, $2x+3=4$, and we solve to obtain $x = \frac{1}{2}$.

1b: Equality of $3^{4x-7} = 3^{6-2x}$ means, because left and right side use the same base 3 for exponentiation, that the exponents must be equal, i.e., $4x-7 = 6-2x$, i.e., $6x = 13$, i.e., $x = 13/6$.

1c: Again, the bases are equal, so the exponents are equal. So, $x^2 - 8 = 2x$, and then $x^2 - 2x - 8 = 0$. We can now factor the left side; $(x-4)(x+2) = 0$, so that $x = 4, -2$.

1d: Once more, equal bases allows us to set the exponents equal. So, $x^3 = x$ so that $x^2 = 1$ and $x^2 - 1 = 0$. We factor to obtain $(x-1)(x+1) = 0$, so that $x = 1, -1$.

1e: We have different bases 2 and 4 and write $4 = 2^2$ to have both bases equal to 2:

$$2^x = (2^2)^{x+5} = 2^{2(x+5)}$$

Now we can equate the exponents: $x = 2x + 10$, i.e., $x = -10$.

1f: We have different bases 9 and 3 and write $9 = 3^2$ to have both bases equal to 3:

$$9^{-4x} - 3^{2x-3} = (3^2)^{-4x} - 3^{2x-3} = 3^{2(-4x)} - 3^{2x-3}.$$

Now we can equate the exponents: $-8x = 2x - 3$, i.e., $10x = 3$, i.e., $x = 3/10 = 0.3$.

Conversion between exponents and logs

Exercise 1:

Rewrite each exponential expression as a log and each log as an exponent.

$$\text{1a: } 10^4 = 10{,}000, \quad \text{1b: } 2^3 = 8, \quad \text{1c: } 1/25 = 5^{-2}, \quad \text{1d: } 9^{1/2} = 3,$$
$$\text{1e: } \log_3 81 = 4, \quad \text{1f: } \log_7 49 = 2, \quad \text{1g: } \log 0.01 = -2, \quad \text{1h: } \ln e = 1$$

Solution to #1:

Recall that $log_a b = c$ can be read as "The exponent that you have to raise a to in order to get b is c".

1a: $\log_{10} 10000 = 4$

1b: $\log_2 8 = 3$

1c: We must rewrite this equation such that it involves logarithmic expressions rather than exponentials. We see that the base is 5 and the exponent is -2, i.e. the result of taking a $\log_5$ logarithm is 2: $\log_5(1/25) = -2$.

1d: $\log_9 3 = \dfrac{1}{2}$

1e: $3^4 = 81$

1f: $7^2 = 49$.

1g: $\log 0.01 \;=\; \log_{10} 0.01 \;=\; -2$ *is a different way of saying that* 0.01 *is the result of raising the base* 10 *to the power of* -2*, i.e.,* $10^{-2} = 0.01$.

1h: $e^1 = e$

Exercise 2: *Without a calculator, find the value of x in each equation.*

$$\text{2a: } \log_5 x \;=\; 3, \quad \text{2b: } \log 1{,}000 \;=\; x, \quad \text{2c: } \log_x 64 \;=\; 3, \quad \text{2d: } \log_{16} x \;=\; 1/4, \quad \text{2e: } \log_x 9 \;=\; 1/2.$$

Solution to #2:

2a: $\log_5 x \;=\; 3$ *means that* $5^3 = x$*, or* $x = 125$.

2b: $\log 1{,}000 \;=\; x$ *means that* $10^x = 1000$*. This is the same as saying that* $10^x = 10^3$*, so that* $x = 3$.

2c: $\log_x 64 \;=\; 3$ *means that* $x^3 = 64$*. Take the cube root of both sides, and* $x = 4$.

2d: $\log_{16} x \;=\; 1/4$ *means* $16^{1/4} \;=\; x$*, i.e.,* $(2^4)^{1/4} \;=\; 2^{1/4\times 4} \;=\; x$*, i.e.,* $x = 2$.

2e: $\log_x 9 \;=\; 1/2$ *means that* $x^{1/2} = 9$*. Square both sides to get* $x = 81$.

Using inverse properties of logs and exponents

Exercise 3:

Simplify each expression.

$$\text{3a: } \log_5 5^{2x+7}, \quad \text{3b: } \log 10^{5x-2}, \quad \text{3c: } \ln e^{5-x},$$
$$\text{3d: } \log_3 9^x, \quad \text{3e: } \log_2 (1/2)^{3x+7}$$

Solution to #3:

3a: Taking the logarithm and exponentiating are inverse operations and the cancel each other but ***only if the same base is used****. This is true for* $\log_5 5^{2x+7}$*:* 5 *is the base in both operations and what remains after cancellation simply is* $2x+7$.

3b: When you see "$\log$*", and the base is not specified, you always assume it is base* 10*. So, in this problem, the* $\log$ *and the* 10 *cancel out, leaving only* $5x - 2$.

3c: When you see "$\ln$*", it is the same as writing "*$\log_e$*", so* $\ln e^{5-x}$ *simplifies to* $5 - x$.

3d: We must adjust bases so they coincide:

$$\log_3 9^x \;=\; \log_3 (3^2)^x \;=\; \log_3 3^{2x} \;=\; 2x.$$

3e: Same technique as 3d and we use the fact that the log of a quotient is the difference of the logarithms:

$$\log_2 (1/2)^{3x+7} \;=\; \log_2 \left(\frac{1}{2^{3x+7}}\right) \;=\; \log_2 1 \;-\; \log_2 2^{3x+7} \;=\; 0 - (3x+7) \;=\; -3x - 7.$$

Using properties of logs

Exercise 4:

Expand each expression into multiple logarithms having single character arguments (if possible).

$$\text{4a: } \log_4(xy), \quad \text{4b: } \log(3x^2), \quad \text{4c: } \log_3(6x+2),$$
$$\text{4d: } \ln\left(\frac{1}{3x}\right), \quad \text{4e: } \ln\left(\frac{5x^4}{2\sqrt{2}}\right), \quad \text{4f: } \log_4\left(\frac{x^3+8x^2+15x}{x+3}\right).$$

Solution to #4:

4a:

$$\log_4(xy) = \log_4 x + \log_4 y$$

4b:

$$\log(3x^2) \;=\; \log 3 + \log(x^2) \;=\; \log 3 + 2\log x.$$

4c: There is no good simplification here.

4d:

$$\ln\left(\frac{1}{3x}\right) = \ln(1) - \ln(3x)$$

4e:

$$\ln\left(\frac{5x^4}{2\sqrt{2}}\right) \;=\; \ln 5 + \ln(x^4) - (\ln 2 + \ln 2^{1/2}) \;=\; \ln 5 + 4\ln x - \ln 2 - \frac{\ln 2}{2}).$$

4f:

$$\log_4\left(\frac{x^3+8x^2+15x}{x+3}\right) = \log_4\left(\frac{x(x^2+8x+15}{x+3}\right) \;=\; \log_4\left(\frac{x(x+3)(x+5)}{x+3}\right)$$
$$= \log_4\Big(x(x+5)\Big) \;=\; \log_4 x \;+\; \log_4(x+5).$$

There are some interesting twists to this solution. First point: We cancelled $x+3$ from both numerator and denominator. We must take note that $x=-3$ does not belong to the domain of this function.
Second point: Arguments to a logarithm must be strictly positive (this also applies to problems D1 - D5): We require that $x>0$ and $x+5>0$, i.e., $x>0$ and $x>-5$, i.e., $x>0$. This requirement ensures that $x \neq -3$ and we see that the explicit requirement that $x=-3$ be excluded is superfluous in this case.

Exercise 5:

Condense each expression into a single logarithm.

$$\text{5a: } \log_5 x + 2\log_5 y, \quad \text{5b: } \log_2 4x - 3\log_2(2y), \quad \text{5c: } (1/2)\ln(x^3) - \ln x.$$

Solution to #5:

5a:

$$\log_5 x + 2\log_5 y \;=\; log_5(x) + log_5(y^2) \;=\; log_5(xy^2)$$

5b:

$$\log_2 4x - 3\log_2(2y) = \log_2(4x) - \log_2((2y)^3) \;=\; log_2(\frac{4x}{(2y)^3})$$

5c:

$$(1/2)\ln(x^3) - \ln x \;=\; \ln(x^3)^{1/2} - \ln x \;=\; \ln(x^{3/2}) - \ln x$$
$$= \ln\left(\frac{x^{3/2}}{x}\right) \;=\; \ln x^{1/2} \;=\; \ln(\sqrt{x}).$$

Exercise 6:

Find a numerical value for each expression.

6a: $(1/3)\log_2 64$, *6b:* $\log 25 + \log 4$, *6c:* $\log_6 9 + \log_6 12 - \log_6 3$.

Solution to #6:

6a:

$$(1/3)\log_2 64 = (1/3)(6) = 2$$

6b:

$$\log 25 + \log 4 \;=\; \log(25 \times 4) \;=\; \log(10^2) \;=\; 2.$$

6c:

$$\log_6 9 + \log_6 12 - \log_6 3 = \log_6(\frac{9 \cdot 12}{3}) = \log_6(36) = 2$$

Using logs to solve exponential equations

Exercise 7:

Solve each expression for x.

7a: $5^{2x} + 15 \;=\; 28$, *7b:* $3(6^{3x+5} - 6) \;=\; 15$, *7c:* $7^x \;=\; 6^{2x+3}$, *7d:* $3^x \;=\; 4^x$ *trick question.*

Solution to #7:

7a: $5^{2x} + 15 = 28$, *so we have* $5^{2x} = 13$. *Let us take* $\log_5$ *of both sides. We choose* $\log_5$, *since we see that* 5^{2x} *has base* 5, *and we know they will cancel out. So,* $\log_5(5^{2x}) = \log_5(13)$, *and then* $2x = \log_5(13)$. *Thus,* $x = \dfrac{\log_5(13)}{2}$.

7b: $3(6^{3x+5} - 6) \;=\; 15$ *means* $6^{3x+5} - 6 \;=\; 5$, *i.e.,* $6^{3x+5} \;=\; 11$. *We take* $\log_6(\cdots)$ *of both sides:*
$3x + 5 \;=\; \log_6 11$, *i.e.,* $3x \;=\; \log_6 11 - 5$, *i.e.,* $x \;=\; (\log_6 11 - 5)/3$.

7c: We take $\log_6(\cdots)$ *of both sides:*

$$\log_6 7^x \;=\; x \cdot \log_6 7 \;=\; \log_6(6^{2x+3}) \;=\; 2x + 3 \quad \textit{means} \quad x \cdot \log_6 7 \;=\; 2x + 3,$$

i.e., $x \cdot \log_6 7 - 2x \;=\; 3$, *i.e.,* $x \;=\; 3/(\log_6 7 - 2)$.

7d (trick question): $3^x = 4^x$ *means* $3^x/4^x = 1$, *i.e.,* $(3/4)^x = 1$.
First solution: Remember what you have learnt about the graph of the function $f(x) = a^x$ *if* $0 < a < 1$ *(in our case* $a = 3/4$*): The function strictly decreases as its argument* x *increases: For any two different numbers* $x_1 < x_2$ *you get*

$a^{x_1} > a^{x_2}$. In particular there is at most one x-value for which $a^x = 1$ and that value is $x = 0$. This is true specifically if $a = 3/4$ and it follows that $x = 0$ is the only value for which $(3/4)^x = 1$, i.e., $3^x = 4^x$.
*Second solution (suggested by Ryan McCullogh): Take on both sides the log to any base $a > 0$, $a \neq 1$ that you are particularly fond of and you get $\log_a(3^x) = \log_a(4^x)$. You can pull the exponent x in front of the logarithm: $x \log_a 3 = x \log_a 4$. This checks out OK only for $x = 0$ **because** for any other x we can divide by x and get $\log_a 3 = \log_a 4$. That's impossible: remember that the graph of the $\log_a$ function strictly increases if $a > 1$ and it strictly decreases if $0 < a < 1$. Either way, $\log_a 3$ and $\log_a 4$ are not the same. We got into this mess because we assumed that $x \neq 0$. In other words, only $x = 0$ is a candidate for $3^x = 4^x$ and indeed we are lucky: $3^1 = 0 = 4^1$.*

Using exponents to solve logarithmic equations

Exercise 8:

Solve each expression for x.

$$\text{8a: } \log_2(3x+2) \;=\; 5, \quad \text{8b: } 14 + 3\log_3 x \;=\; 20,$$
$$\text{8c: } \log_4(2x+5) - \log_4(x-1) \;=\; 2, \quad \text{8d: } 2\ln(x-3) - \ln(21-2x) \;=\; 0.$$

Solution to #8:

8a: $\log_2(3x+2) \;=\; 5$ means that $2^5 = (3x+2)$. So, $32 = (3x+2)$, and then $x = 10$.

8b: First, simplify. $14 + 3\log_3 x \;=\; 20$ turns into $3\log_3 x = 6$. So, $\log_3 x = 2$. Thus, $3^2 = x$, and $x = 9$.

8c: First, condense all the logs into one piece. $\log_4(2x+5) - \log_4(x-1) \;=\; 2$ means that $\log_4 \dfrac{2x+5}{x-1} = 2$. Raise 4 to both sides to cancel out $\log_4$. So, $4^{\log_4(\frac{2x+5}{x-1})} \;=\; 4^2 \;=\; 16$, and so $\dfrac{2x+5}{x-1} = 16$. Thus, $2x+5 = 16x-16$, and so $14x = 21$, and $x = 3/2$.

8d: $2\ln(x-3) - \ln(21-2x) \;=\; 0$ means $\ln(x-3)^2 \;=\; \ln(21-2x)$. We do $e^{(\cdots)}$ on both sides: $(x-3)^2 \;=\; 21-2x$, i.e., $x^2 - 6x + 9 + 2x = 21$, i.e., $x^2 - 4x - 12 \;=\; 0$. We factor $x^2 - 4x - 12 \;=\; (x-6)(x+2) \;=\; 0$ which means $x = 6$ or $x = -2$.
We are not allowed to use the solution $x = -2$: If we do then $\ln(x-3) = \ln(-5)$ and a logarithm function must never have a non-positive argument!

Using the change of base formula

Exercise 9:

Rewrite each expression as either a single natural log or exponent.

$$\text{9a: } \log_5 x, \quad \text{9b: } 3\log_7 x, \quad \text{9c: } 4^x, \quad \text{9d: } 6^{3x+2}.$$

Solution to #9:

9a:

$$\log_5 x \;=\; \ln x/\ln 5 \;=\; \ln x^{1/\ln 5}.$$

9b:

$$3\log_7 x \;=\; 3\frac{\ln x}{\ln 7} \;=\; \frac{3}{\ln 7}\ln x \;=\; \ln x^{3/\ln 7}.$$

9c:

$$4^x \;=\; e^{\ln(4^x)} \;=\; e^{x\ln(4)}.$$

9d:

$$6^{3x+2} \;=\; e^{\ln(6^{3x+2})} \;=\; e^{(3x+2)\ln 6}.$$

Challenge problem

Exercise 10:

$$\textit{Solve for } x: \quad \frac{\ln 8}{\ln x} \;=\; \frac{\ln x^9}{\ln 8}$$

Solution to #10:

The trick is to jumble this equation around such that you deal with an equation of squares:

$$\begin{aligned}
(\ln 8)(\ln 8) \;&=\; (\ln x^9)(\ln x) \;=\; 9(\ln x)(\ln x), \;\textit{i.e.},\\
(\ln 8)^2 \;&=\; (\pm 3)^2(\ln x)^2 \;=\; (\pm 3\ln x)^2, \;\textit{i.e.},\\
(\ln 8) \;&=\; (\pm 3)(\ln x), \;\textit{i.e.},\\
(\ln x) \;&=\; (\pm 1/3)(\ln 8) \;=\; \ln(8^{\pm 1/3}), \;\textit{i.e.},\\
x \;&=\; 8^{1/3} \;=\; 2 \quad \textit{or} \quad x \;=\; 8^{-1/3} \;=\; 1/2
\end{aligned}$$

Just to make sure all is OK, you could plug those two solutions into the expression $(\ln x^9)(\ln x)$*. This better be equal to* $(\ln 8)^2$*! I'll demonstrate this for* $x = 2$*:*
The Change of Base formula gives us

$$\begin{aligned}
\ln x^9 \;&=\; \ln 2^9 \;=\; (\log_2 2^9)/(\log_2 e) \;=\; 9/\log_2 e\\
\textit{and} \quad \ln x \;&=\; \ln 2 \;=\; (\log_2 2)/(\log_2 e) \;=\; 1/\log_2 e\\
\textit{i.e.}, \quad (\ln x^9)(\ln x) \;&=\; 9/(\log_2 e)^2.
\end{aligned}$$

On the other hand, again making a change of base from e *to* 2*,*

$$\begin{aligned}
\ln 8 \;&=\; \ln 2^3 \;=\; (\log_2 2^3)/(\log_2 e) \;=\; 3/\log_2 e\\
\textit{i.e.}, \quad (\ln 8)^2 \;&=\; (3/\log_2 e)^2 \;=\; 9/(\log_2 e)^2
\end{aligned}$$

and that is the same result as what we got for $(\ln x^9)(\ln x)$*.*