

## Sec 29 - Exercises

2a) Maximize  $f(x,y) = 2xy$  subj to  $x+y=12$

$$F(x,y,\lambda) = 2xy + \lambda(x+y-12)$$

$$F_x = 2y + \lambda = 0$$

$$F_y = 2x + y = 0$$

$$F_\lambda = x+y-12 = 0$$

$$F_x - F_y = 2y - 2x = 0$$

$$y = x$$

Into constraint (or  $g(x,y)=0$ )

$$x+x-12=0$$

$$2x = 12, \boxed{x=6}$$

$$\frac{\partial f}{\partial x} = 2y, \quad \lambda \frac{\partial g}{\partial x} = \lambda \cdot 1$$

$$\frac{\partial f}{\partial y} = 2x, \quad \lambda \frac{\partial g}{\partial y} = \lambda \cdot 1$$

$$g(x,y)=0$$

Setting partials equal.

$$\begin{aligned} 2y &= \lambda \\ 2x &= \lambda \end{aligned} \rightarrow \boxed{y=x}$$

$$x+x=12$$

$$2x=12, \boxed{x=6, y=6}$$

2c. This one is a bit different, since if you eliminate  $\lambda$ , you get ~~cancel~~ no  $x$  or  $y$  either! Shown below using Dook's way:

$$f(x,y) = x^2 - y^2; \quad x^2 + y^2 = 4 \Rightarrow g(x,y) = x^2 + y^2 - 4$$

$$F = x^2 - y^2 + \lambda(x^2 + y^2 - 4)$$

$$F_x = 2x + 2\lambda x = 0 \rightarrow 2x(1 + \lambda) = 0$$

$$F_y = -2y + 2\lambda y = 0 \rightarrow -2y(1 - \lambda) = 0$$

$$F_\lambda = x^2 + y^2 - 4 = 0 \rightarrow x^2 + y^2 - 4 = 0$$

Consider  $F_x$  and  $F_y$ . They basically say  $A=0$   
so either  $A=0$  or  $B=0$ . That is,

$$F_x = 2x(1 + \lambda) = 0 \rightarrow \begin{cases} 2x = 0 \\ 1 + \lambda = 0 \end{cases} \quad \textcircled{R} \quad \boxed{\lambda = -1}$$

$\lambda = -1$  isn't relevant to the constraint, but substituting  $x=0$  into  $x^2 + y^2 = 4$  gives two possible  $y$  values:  $0 + y^2 = 4 \rightarrow \boxed{y = \pm 2}$

Test the two points  $(0, 2)$  and  $(0, -2)$  into  $f(x,y)$  to see which gives the maximum.

$$f(0,2) = 0 - 2^2 = -4, \quad f(0,-2) = 0 - (-2)^2 = -4$$

So far, the two solns. give the same negative answer.

Go back to the partials, this time,  $F_y$ :

$$F_y = -2y(1-\lambda) = 0 \rightarrow -2y = 0 \text{ or } 1-\lambda = 0$$
$$\boxed{y=0} \qquad \lambda = 1$$

Substitute  $y=0$  into the constraint:

$$x^2 + y^2 = 4 \rightarrow x^2 + 0^2 = 4 \rightarrow \boxed{x = \pm 2}$$

Test the two pts  $(2,0)$  and  $(-2,0)$  into  $f(x,y)$  and compare to previous answers.

$$f(2,0) = 2^2 - 0 = 4, \quad f(-2,0) = (-2)^2 - 0 = 4$$

Hence, both  $(2,0)$  and  $(-2,0)$  give the maximum  $f(x,y)$ . And  $(0,2)$  and  $(0,-2)$  give the minimum.

$$2d) \quad f(x,y) = x+y-x^2-y^2, \quad \begin{cases} x+2y=6 \\ g(x,y)=c \end{cases}$$

Let's do the video way; (keep  $x+2y=6$  as is.)

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \quad \rightarrow \quad 1-2x = \lambda \cdot 1 \quad \textcircled{1}$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \quad \rightarrow \quad 1-2y = \lambda \cdot 2 \quad \textcircled{2}$$

$$g(x,y)=c \quad \rightarrow \quad x+2y=6 \quad \textcircled{3}$$

Eliminate  $\lambda$  from first two:

$$\textcircled{1} \quad 1-2x = \lambda \quad \xrightarrow[\textcircled{2}]{\substack{\text{sub } \lambda \\ \text{into}}} \quad 1-2y = ((-2x) \cdot 2)$$

$$\begin{array}{l} \xrightarrow{\text{Solve for}} \\ y \text{ in terms} \\ \text{of } x \end{array} \quad \begin{array}{l} -2y = 2-4x-1 \\ -2y = 1-4x \\ \boxed{y = \frac{1-4x}{-2}} \end{array}$$

$$\xrightarrow[\text{into}]{\text{Sub } y} \quad x + 2 \left( \frac{1-4x}{-2} \right) = 6 \quad \begin{array}{l} \xrightarrow{\text{sub } x} \\ \text{constraint} \end{array}$$

constraint

$$x+2y=6$$

$$-2x + 2 - 8x = -12$$

$$-10x = -14$$

$$x = 14/10 = 7/5 = x \quad \boxed{y = \frac{23}{10}}$$

$$\frac{7}{5} + 2y = 6$$

The value of  $f(x,y)$  is  $f\left(\frac{7}{5}, \frac{23}{10}\right) = -3 \frac{55}{100}$

or  $-3.55$

#2f.  $f(x,y) = 2x^2 + y^2 - 4y$  subj to  $x^2 + y^2 = 1$

Text way:  $F(x,y,\lambda) = 2x^2 + y^2 - 4y + \lambda(x^2 + y^2 - 1)$

$$F_x = 4x + 2x\lambda = 0$$

$$F_y = 2y - 4 + 2y\lambda = 0$$

$$F_\lambda = x^2 + y^2 - 1 = 0$$

This one is much like 2c,  
where elimination of  $\lambda$   
doesn't work.

But if we factor  $F_x$  into  $2x(2 + \lambda) = 0$

so  $2x = 0$  or  $2 + \lambda = 0$ , so  $\underline{x=0}$  or  $\underline{\lambda = -2}$   
could be used to find possible soln.  $(x,y)$ .

If we go through  $\lambda$ , we have to get a new eqn in  $(x,y)$  and set up a system of two eqns with this new one and  $F_\lambda$ . Ugh. — So let's go through  $x=0$  & constraint:

$$x^2 + y^2 = 1$$

$$y = \pm 1$$

$$x^2 + y^2 = 1$$

graph

Points:  $(0,1), (0,-1)$

$$f(0,1) = -3 \text{ min}$$

$$f(0,-1) = 5 \text{ max}$$

Word problems:

3.  $x, y > 0$ ,  $x+y = 35$ , maximize  $x^2y$

Video way:  $f(x,y) = x^2y$ ,  $g(x,y) = c$   
 $\hookrightarrow x+y = 35$

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2xy, & \frac{x\partial g}{\partial x} &= \lambda \cdot 1 \\ \frac{\partial f}{\partial y} &= x^2 & \frac{\lambda \partial g}{\partial y} &= \lambda \cdot 1 \\ g &= x+y = 35 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 2xy &= \lambda \\ x^2 &= \lambda \\ \hline 2xy - x^2 &= 0 \\ y &= \frac{x^2}{2x} = \frac{x}{2} \end{aligned}$$

Sub  $y = \frac{x}{2}$  into  $x+y=35$

$$x + \frac{x}{2} = 35 \rightarrow \frac{3x}{2} = 35 \rightarrow \boxed{x = \frac{70}{3}}$$
$$y = 35 - \frac{70}{3}$$
$$\boxed{y = \frac{35}{3}}$$

#5:

Area: top + bottom =  $2xy$  @  $.20/\text{ft}^2$

• sides =  $2x^2 + 2xy$  @  $.10/\text{ft}^2$

Cost  $C(x,y) = .2(2xy) + .1(2x^2 + 2xy)$  to minimize

$$V = xy^2 = 2 \text{ constraint}, \quad g(x,y) = x^2y^2 - 2 = 0$$

$$\frac{\partial C}{\partial x} = \lambda \frac{\partial g}{\partial x} \rightarrow .4y + .4x + .2y = \lambda \cdot 2xy \quad (1)$$

$$\frac{\partial C}{\partial y} = \lambda \frac{\partial g}{\partial y} \rightarrow .4x + .2x = \lambda x^2 \quad (2)$$

$$x^2y = 2 \quad (3)$$

$$(2) .6x = \lambda x^2 \rightarrow \lambda = \frac{.6x}{x^2} = \frac{.6}{x} \xrightarrow{\text{into } (1)}$$

$$.4y + .4x + .6y = \frac{.6}{x} \cdot 2xy \quad \xrightarrow{\text{into } (3)}$$

$$1y + .4x = 1.2y$$

$$.4x = .2y$$

$$x = \frac{y}{2}$$

$$x^2y = 2$$

$$\frac{y^2}{4} \cdot y = 2$$

$$y^3 = 8$$

$$\boxed{y = 2}$$

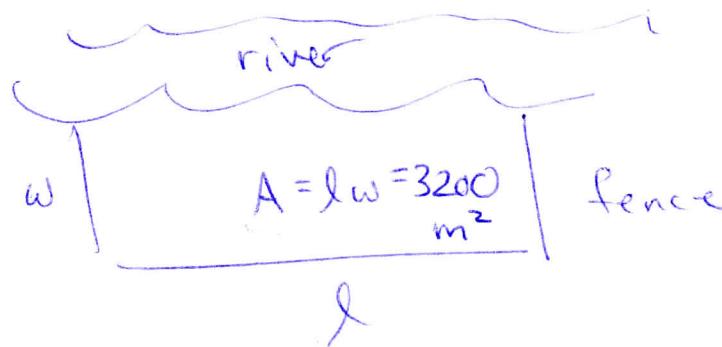
$$\boxed{x^2y = 2}$$

$$2x^2 = 2$$

$$\boxed{x = 1}$$

The last part was all substitution into the constraint. But eliminating  $\lambda$  in (2) to sub into (3) was tricky!

7.



Minimize length of fence:  $P(l, w) = 2w + l$   
 Constraint:  $A = lw = 3200 \Rightarrow g(l, w) = lw - 3200$

$$F(l, w, \lambda) = 2w + l + \lambda(lw - 3200)$$

$$\textcircled{1} \quad F_l = 1 + \lambda w = 0$$

$$\lambda = -\frac{1}{w} \quad \textcircled{1} \xrightarrow{\text{sub into}}$$

$$\textcircled{2} \quad F_w = 2 + \lambda l = 0$$

$$2 + \frac{-1}{w} \cdot l = 0 \quad \textcircled{2}$$

$$\textcircled{3} \quad F_\lambda = lw - 3200 = 0$$

$$2w = +l \\ w = +\frac{l}{2}$$

$$l\left(+\frac{l}{2}\right) - 3200 = 0$$

$$\lambda \xrightarrow{\text{sub into } \textcircled{3}}$$

$$+\frac{l^2}{2} = 6400$$

$$l = 80 \text{ m}, \quad \text{use } \sqrt{3200}$$

$$w = \frac{l}{2} = \frac{80}{2} = 40 \text{ m}$$

$$l = 80 \text{ m}$$

$$w = 40 \text{ m}$$

$$P = 2(80) + 40 = 200 \text{ m fence}$$