Applications of the Derivative: Optimization

3.3

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As we mentioned in the introduction to this chapter, derivatives can be used to determine the optimum price for a product; that is, the price that produces the most profit for a business. We looked at the example of Sally, who owns an online business that sells hard drives. By applying a derivative to a function that relates the price of hard drives, the number sold, the cost of the hard drive to the retailer, and the number of hard drives purchased by the retailer, Sally can determine how best to price her product. These sorts of calculations are done on a constant basis by big Internet retailers as they seek to maximize their profits.

The process of determining the maximum or minimum values of a function is called **optimization**. Using the differentiation techniques discussed in Sections 3.1 and 3.2, we will see how the derivative is applied to optimize functions that model practical applications from business, economics, medicine, and other fields.

A Strategy for Solving Optimization Problems

We will now consider how differential calculus can be used to solve optimization problems. These are problems in which we are interested in maximizing or minimizing a particular quantity. For example, we may be interested in knowing the sales price of an item that will maximize the profit on that item, or how many days after an insecticide is applied to a population of insects, the population will be at its minimum.

A strategy that is helpful in solving optimization problems follows.

1. Draw a figure when one is appropriate.
2. Assign a variable to each quantity mentioned in the problem.
3. Select the quantity that is to be optimized and construct a function that relates it to some or all of the other quantities.
4. Since all the rules of differentiation deal with functions of only one variable, the function must involve only one variable. If the function constructed in step 3 involves more than one variable, use the information contained in the problem to eliminate variables until you have a function of only one variable.
5. Use differentiation to find the critical values of the function and apply the test for absolute extrema. Disregard all answers that are not relevant to the situation.

Applying the Strategy: Examples

We will illustrate the optimization strategy with five examples. We will show the first example with great detail and then include less detail where appropriate in the remaining examples.

VIDEO EXAMPLES

SECTION 3.3

Example 1 A rectangular region along the bank of a straight section of a river is to be enclosed with 80 feet of fencing. Only three sides need to be fenced since the river will bound the other side. What are the dimensions that will produce the greatest enclosed area?

Solution We will apply the optimization strategy.

1. We will construct a figure since one is appropriate. (See Figure 1.)

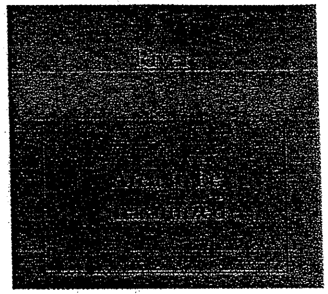


Figure 1

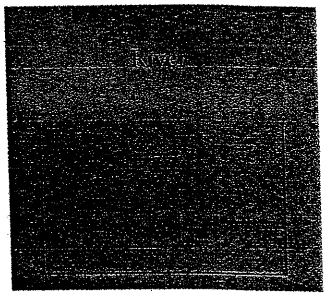


Figure 2

2. The quantities mentioned in this problem are the area, the dimensions of the region, and the amount of fencing available. We will label each with a variable and then affix the appropriate labels to our picture, as shown in Figure 2.
   $A =$ the area of the rectangular region
   $l =$ the length of the rectangular region
   $w =$ the width of the rectangular region
   $P =$ the perimeter of the rectangular region (that is, the amount of fencing available)
3. The quantity to be maximized is the area, so we need to construct an area function. Since the region is rectangular, we will use the formula for the area of a rectangle.

   Maximize: $A(l, w) = lw$.
   (Remember how functions are read: $A(l, w)$ means that the area A depends on both the length l and the width w.)
4. This function involves two variables, l and w; we need to eliminate one. To do so, we will use the information contained in the problem to express l in terms of w or w in terms of l. The problem restricts the amount of fencing to 80 ft, which means that the *perimeter*, P, of the fenced region is 80 ft. The perimeter of this shape is the sum of the lengths of the three fenced sides, so

$$P = w + l + w = 80$$
$$= l + 2w = 80$$

   Thus, $l + 2w = 80$, and we can solve for either l or w. (We can avoid fractions by solving for l. Avoiding fractions may not always be possible.)

$$l + 2w = 80$$
$$l = 80 - 2w$$

Now, replacing l in the function $A(l, w) = lw$ with $80 - 2w$ gives us the function of the one variable w.

$$A(w) = (80 - 2w)w$$
$$A(w) = 80w - 2w^2$$

A negative area is not acceptable; thus, solving $A(w) \geq 0$ (by sign charting) gives us $0 \leq w \leq 40$. Therefore, $A(w)$ is defined over a closed interval $[0, 40]$.

5. To find the critical values and then the absolute extreme values, we first find the first derivative.

$$A'(w) = 80 - 4w$$

a. We then ask: Where is $A'(w) = 0$?

$$A'(w) = 0$$
$$80 - 4w = 0$$
$$-4w = -80$$
$$w = 20$$

So, $A'(w) = 0$ when $w = 20$.

b. We then ask: Where is $A'(w)$ undefined? $A'(w)$ is never undefined since it is a polynomial function.

Thus, the only critical value is $w = 20$. Since we are interested only in the absolute extrema, we only need to evaluate $A(w)$ at 0, 20, and 40, to see which produces the greatest output (area).

$$A(0) = 0,$$
$$A(20) = 80(20) - 2(20)^2 = 800,$$
$$A(40) = 80(40) - 2(40)^2 = 0$$

Now, we conclude that 20 produces the absolute maximum value of the function. We can also use $l = 80 - 2w$ to find that $l = 80 - 2(20) = 40$.

Interpretation

The dimensions that produce the maximum enclosed area for 80 feet of fencing are a length of 40 ft and a width of 20 ft. The graph of $A(w) = 80 - 2w^2$ appears in Figure 3 and reinforces the fact that the absolute maximum area occurs when the width is 20 feet.

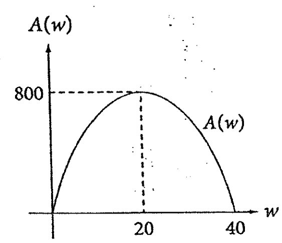


Figure 3

Example 3 Sally owns an online business and wants to know how to price her inventory of 3-terabyte hard drives in order to maximize the profit she makes from their sale. Currently, Sally sells 20 hard drives a week for \$400 each (which cost the store \$200 each). She estimates that for each \$10 reduction in price, she can sell two more drives each week. Find the price of each drive that will produce the maximum profit for her business.

Solution

1. A picture is not appropriate here so we will not construct one.
2. We will introduce some variables to represent the quantities mentioned in the problem.
   r = the number of \$10 reductions in price
   p = the price of each hard drive after a reduction
   n = the number of hard drives purchased and sold
   R = the revenue produced by all the sale of n hard drives
   C = the cost to the store of n hard drives
   P = the profit realized by the sale of n hard drives
3. We are asked to find the price that maximizes the profit, so we need to construct a profit function. Since profit equals revenue minus cost, we have

$$P = R - C$$

But this is a function of two variables. Since revenue is price times the number sold, and cost is the individual cost times the number purchased,

$$R = \underbrace{(400 - 10r)}_{\text{Price}} \cdot \underbrace{n}_{\text{Number sold}} \quad \text{and} \quad C = \underbrace{200}_{\text{Cost}} \cdot \underbrace{n}_{\text{Number purchased}}$$

4. That is, $R = (400 - 10r)n$ and $C = 200n$. The expression for R comes from the fact that the sales price is \$400 minus the \$10 for every reduction; thus, \$400 − $10r$. But also, the number of drives sold n is 20 plus 2 times the number of \$10 price reductions, that is, $n = 20 + 2r$. Hence,

$$P(r) = (400 - 10r)(20 + 2r) - 200(20 + 2r)$$
$$= 4{,}000 + 200r - 20r^2$$

and now the profit, P, depends only on the number, r, of \$10 price reductions.

5. $P'(r) = 200 - 40r$.
   The critical value is $r = 5$, and the first derivative test shows this to produce a relative maximum.

Interpretation

The best price per drive is \$400 − 10(5) = \$350. This would produce sales of $n = 20 + 2(5) = 30$ drives per week, and a maximum profit (per week) of

$$P(5) = 4{,}000 + 200(5) - 20(5)^2 = \$4{,}500$$

These require the method outlined at the start of Sec 23 in our book.

16. **Business: Airline Revenue** A charter flight club charges its members \$250 per year. But for each new member above a membership of 65, the charge for all members is reduced by \$2 each. What number of members leads to a maximum revenue?

17. **Business: Agricultural Yield** An orchard presently has 30 trees per acre. Each tree yields 480 peaches. It is determined that for each additional tree that is planted per acre, the yield will be reduced by 12 peaches per tree. How many trees should be added per acre to maximize the yield?

18. **Business: Demand and Revenue** Demand for an electric fan is related to its selling price p (in dollars) by the equation

$$n = 2{,}880 - 90p$$

where n is the number of fans that can be sold per month at a price p. Find the selling price that will maximize the revenue received.

19. **Business: Packaging** Postal regulations specify that a parcel sent by parcel pos may have a combined length and girth (distance around) not exceeding 10(inches. If a rectangular package has a square cross section, find the dimension: of the package with the largest volume that may be sent through the mail.

20. **Business: Packaging Costs** The More Beef Company requires its corned bee hash containers to have a capacity of 64 cubic inches. The containers are in th shape of right circular cylinders. Find the radius and height of the containe that can be made at a minimum cost, if the tin alloy for the side and botton costs 4 cents per square inch and the aluminum for the pull-off lid costs : cents per square inch.

21. **Business: Maximum Revenue** A tool company determines that it can achiev 500 daily rentals of jackhammers per year at a daily rental fee of \$30. For eac \$1 increase in rental price, 10 fewer jackhammers will be rented. What rent price maximizes revenue?

22. **Business: Motel Revenue** A motel finds that it can rent 200 rooms per day if charges \$80 per room. For each \$5 increase in rental rate, 10 fewer rooms wi be rented per day. What room rate maximizes revenue?

Using Technology Exercises

23. Find the value of x that minimizes the function

$$f(x) = x^{5/3} - 5x^{2/3} + 3$$

on the interval [0, 5] and specify the value of x at which it occurs.

24. The concentration, C, of a drug in the bloodstream t hours after it has bee administered is approximated by the function

$$C(t) = \frac{12t}{t^{3.2} + 24} \qquad t \geq 0$$

When is the concentration the greatest and what is that concentration?

Skills Practice

The first 5 are simple optimization, in which the model is the given fcn. (similar to earlier sections — just differentiate + solve)

1. **Health Science: Drug Concentration** The concentration of a drug in the bloodstream $C(t)$ at any time t, in minutes, is described by the equation

$$C(t) = \frac{100t}{t^2 + 16}$$

where $t = 0$ corresponds to the time at which the drug was swallowed. Determine how long it takes the drug to reach its maximum concentration.

2. **Health Science: Flu Outbreak** According to a model developed by a public health group, the number of people $N(t)$, in hundreds, who will be ill with the Asian flu at any time t, in days, next flu season is described by the equation

$$N(t) = 90 + \frac{9}{4}t - \frac{1}{40}t^2 \quad 0 \leq t \leq 120$$

where $t = 0$ corresponds to the beginning of December. Find the date when the flu will have reached its peak and state the number of people who will have the flu on that date.

3. **Business: Heating Costs** A homeowner wishes to insulate his 1,500-square-foot attic. The total cost $C(r)$ for r inches of insulation and the heating costs for that amount of insulation over the next 10 years is given by

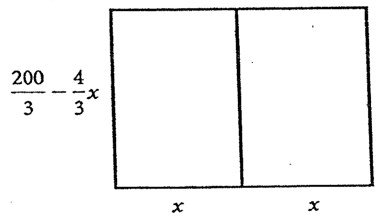
$$C(r) = 120r + \frac{4{,}320}{r}$$

How many inches of insulation should be placed in the attic if the total cost is to be minimized?

Source: CPS Energy

4. **Business: Maximization of Space** A rancher has 200 feet of fencing to enclose two adjacent rectangular corrals, as shown in the following figure. The equation describing the enclosed area is

$$A(x) = 2x\left(\frac{200}{3} - \frac{4}{3}x\right)$$



To enclose the maximum area, what should be the dimensions of each corral?

5. **Business: Revenue** Suppose a baby food company has determined that its total revenue R for its food is given by

$$R = -x^3 + 63x^2 + 1{,}200x$$

where R is measured in dollars and x is the number of units (in thousands) produced. What production level will yield a maximum revenue?