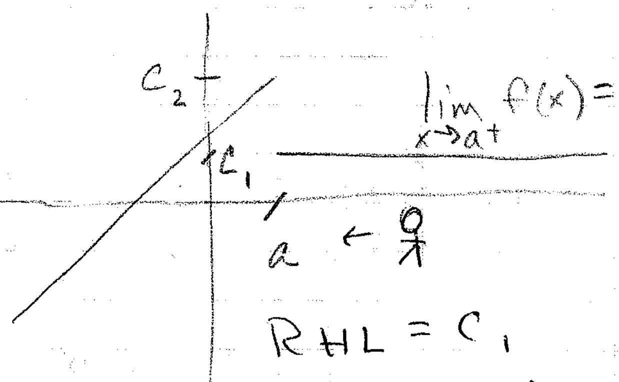
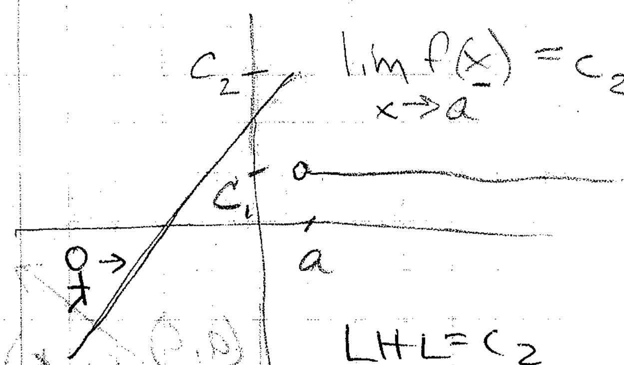
One-sided limits

Sometimes a limit exists from one side of $x = a$ that is not the same as the limit from the other side.

Right hand limit (RHL) $\neq$ Left hand limit (LHL)



$RHL = c_1$



$LHL = c_2$

$\lim_{x \to a} f(x)$ DNE but $\boxed{f(a) = c_2}$

If the RHL = LHL then we say the limit exists $\lim_{x \to a} f(x) = L$

Above $\lim_{x \to a^+} f(x) = c_1 \neq c_2 = \lim_{x \to a^-} f(x)$

So RHL $\neq$ LHL

and hence $\lim_{x \to a} f(x)$ DNE

Limits

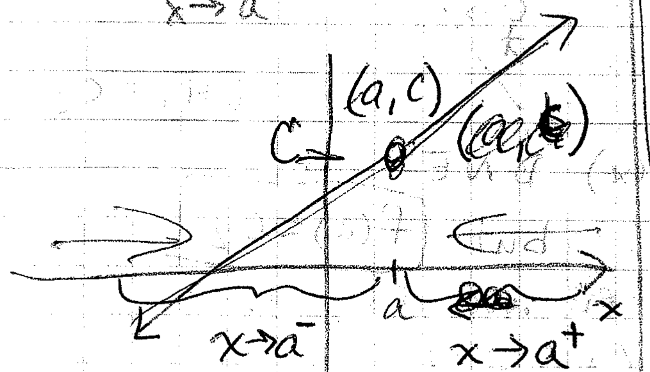
Finite

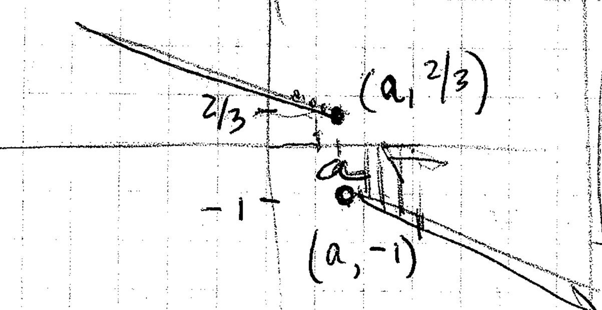
$\lim_{x \to a} f(x) = L$

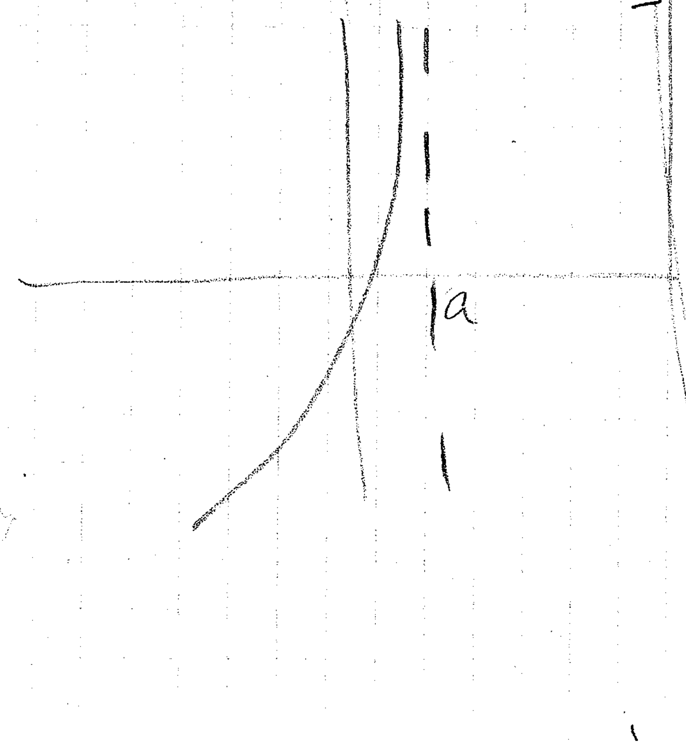
finite value of x

or

$\lim_{x \to a} f(x)$ DNE



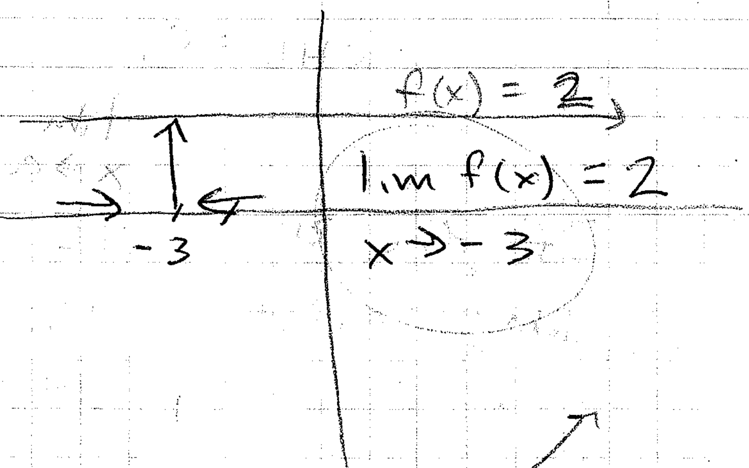




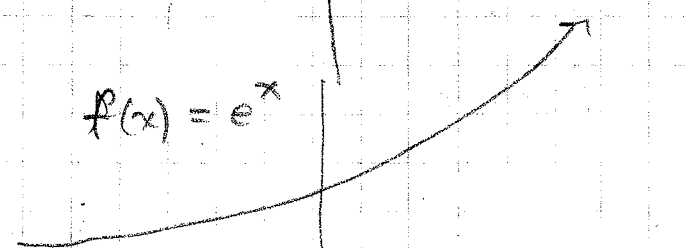
Infinite

$\lim_{x \to a} f(x) = \pm\infty$ or DNE

$\lim_{x \to \infty} f(x) = 2$, $\lim_{x \to -\infty} f(x) = 2$



$f(x) = e^x$

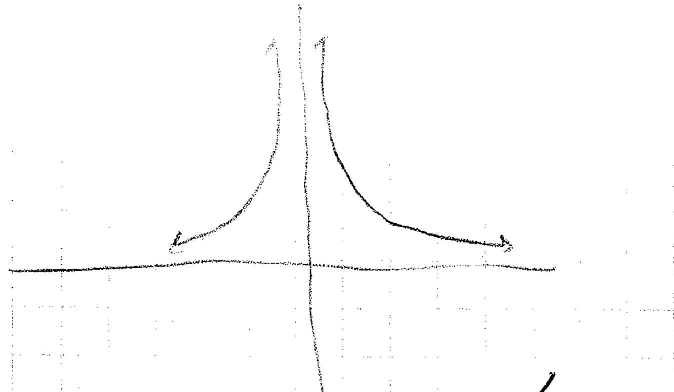


$\lim_{x \to \infty} e^x = \infty$ or DNE

$\lim_{x \to -\infty} e^x = 0$

Ex 6.13 $\quad f(x) = \frac{1}{x^2}$

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$



LHL as $x \to 0$ is $+\infty$
RHL as $x \to 0$ is $+\infty$

* Ex 6.14 $\quad f(x) = \frac{3x+6}{x^2-3x-10}$

deg numerator $<$ deg denominator

HA: $y = 0$

$$\lim_{x \to -2} \frac{3x+6}{x^2-3x-10} = \lim_{x \to -2} \frac{3\cancel{(x+2)}}{\cancel{(x+2)}(x-5)}$$

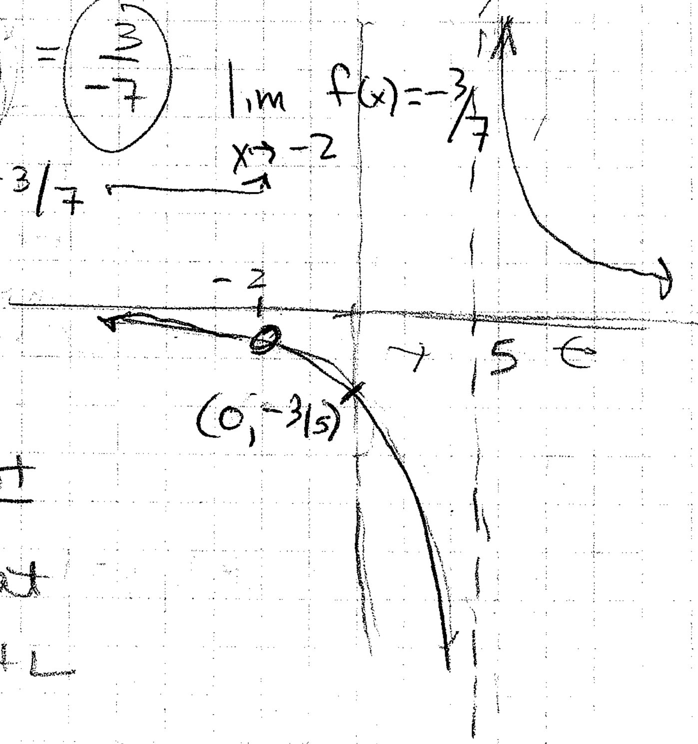
Dom: $x \neq -2, 5$

$$\lim_{x \to -2^-} \frac{3}{x-5} = \lim_{x \to -2^+} \frac{3}{x-5} = \frac{3}{-7}$$

$$\lim_{x \to -2} f(x) = -3/7$$

LHL = RHL = $-3/7$

$$\lim_{x \to 5} \frac{3}{x-5} = ?$$



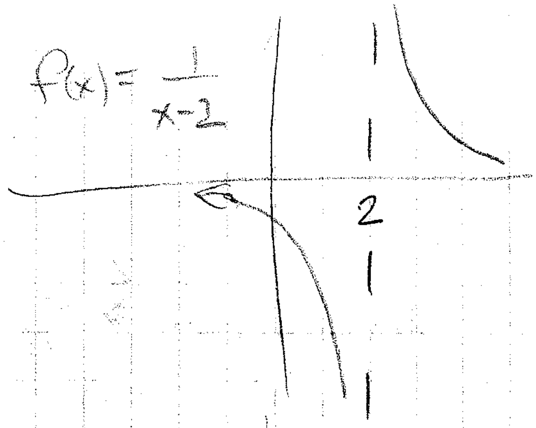
[When you get $\frac{\text{const}}{0}$ you need to look at the LHL and RHL since this indicates an asymptote. This is why knowing your basic graphs makes limits much easier to determine]

$$\lim_{x \to 5^-} \frac{3}{x-5} = -\infty \qquad \lim_{x \to 5^+} \frac{3}{x-5} = \infty$$

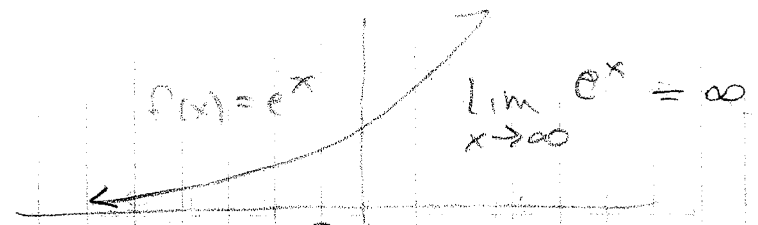
LHL $\neq$ RHL so $\lim_{x \to 5} \frac{3}{x-5}$ DNE

Infinite limits (not to be confused with limits at ∞)

$f(x) = \frac{1}{x-2}$



$f(x) = e^x$



$\lim_{x \to \infty} e^x = \infty$

$\lim_{x \to -\infty} e^x = 0$

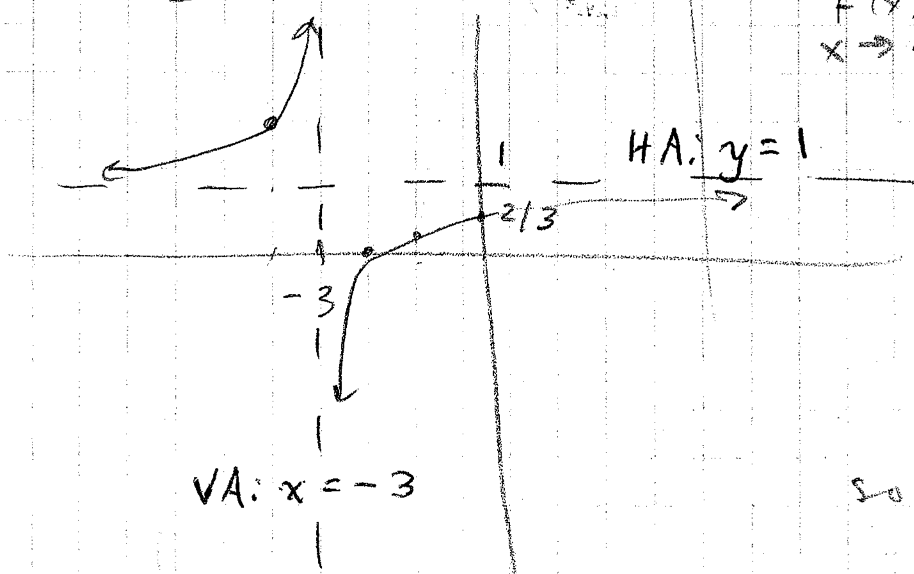
$\lim_{x \to 2^-} \frac{1}{x-2} = -\infty$

$\lim_{x \to 2^+} \frac{1}{x-2} = \infty$

So $\lim_{x \to 2} \frac{1}{x-2}$ DNE

This graph shows how the exponential fcn goes to ∞ as $x \to \infty$ and goes to zero as $x \to -\infty$. We haven't done anything with $\lim_{x \to \infty} f(x)$ yet. However, you may remember that this is the way to inspect for horizontal asymptote.

Ex 6.12



$f(x) = \frac{x+2}{x+3}$
$x \to -3$

$\lim_{x \to -3^+} f(x) = -\infty$

$\lim_{x \to -3^-} f(x) = \infty$

So $\lim_{x \to -3} f(x)$ DNE

Note: The HA of a rational fcn. whose numerator + denominator are of the same degree is the line $y =$ ratio of leading coefficients of numerator + denominator

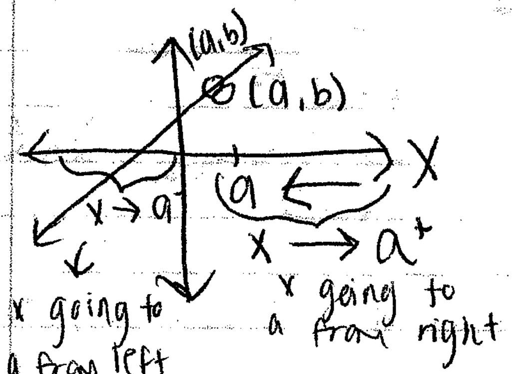
2/10

Sec. 6 – A summary

• The "Limit" of a function as x approaches a value "a" is written limit f(x). If the limit exists, call it "L", then we write limit f(x) = L x→a. This limit exists only if limit as x approaches the value "a" from the left exists and is equal to the limit as x approaches "a" from the right. This is written as $\boxed{\lim_{x\to a^-} f(x)}$ = left hand limit (LHL), $\lim_{x\to a^+} f(x)$ = right hand limit (RHL)

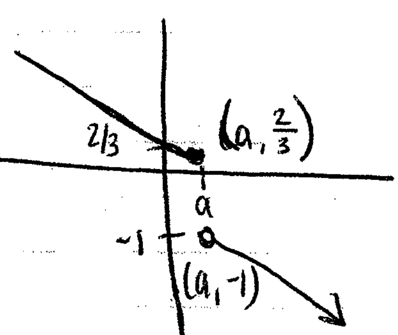
and if LHL = RHL, then limit f(x) exsists call it L. x→a from both sides

Note: limit f(x) may be a finite value or it may be "unbanded", that is, infinitely positive or negative



f(a) Daes not exist, but
limit f(x) = b x→a⁺
$\lim_{x\to a^-} f(x) = b$

• so limit f(x) = b, x→a from both sides
• b is the limit



$\lim f(x) = \frac{2}{3}$ $x \to a^-$

$f(a) = 2/3$

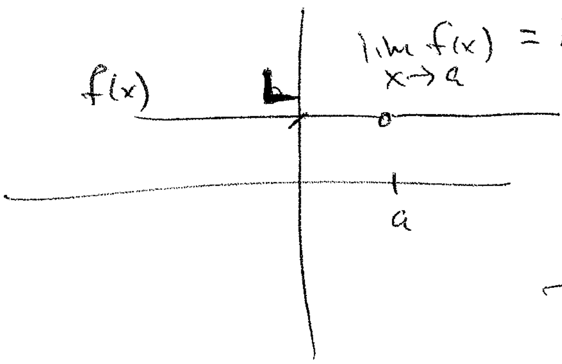
$\lim_{x\to a^+} f(x) = -1$

$\lim_{x\to a} f(x)$ DNE does not exist.

2/11

Section 9 - Continuity of $f(x)$

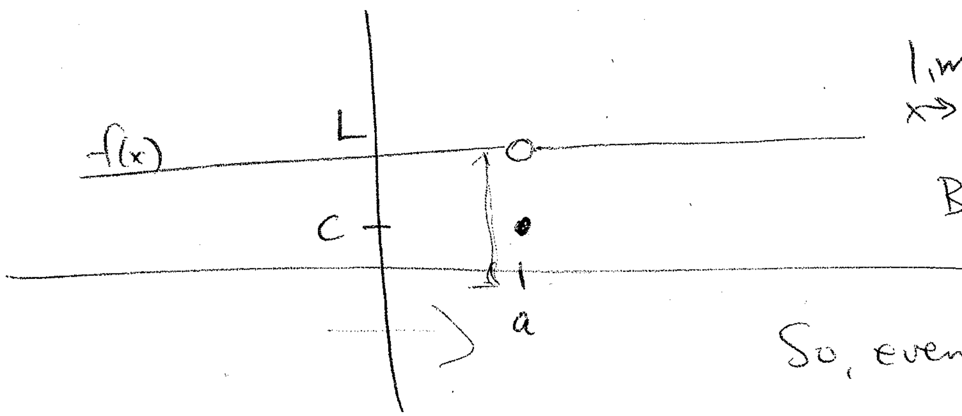
Def A function is said to be "continuous at $x = a$" if $\lim_{x \to a} f(x)$ exists and is equal to $f(a)$ itself



$\lim_{x \to a} f(x) = L$ but $f(a) \neq L$

because $f(a)$ has no value

Therefore $f(x)$ is not cts. at $x = a$.



$\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

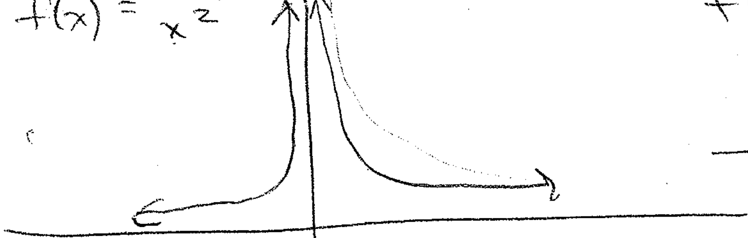
But $f(a) = C$

So, even though $\lim_{x \to a} f(x)$ exists, and it's L,

$f(a) = C \neq L$, so

$f(x)$ is not cts. at $x = a$

$f(x) = \frac{1}{x^2}$



LHL = RHL = ∞

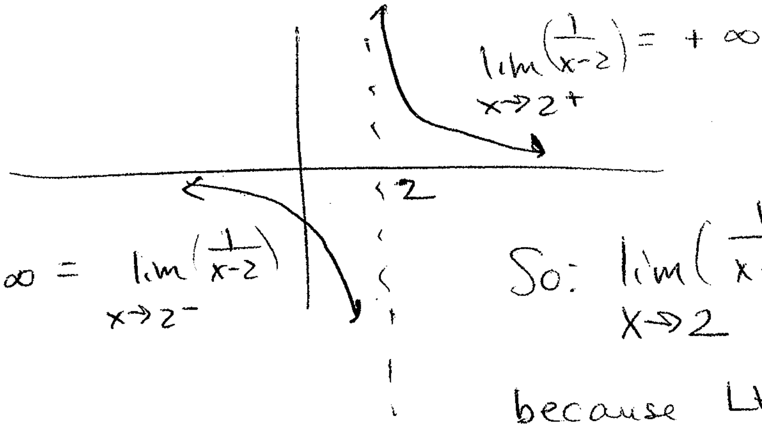
But $f(x)$ is not cts at $x = 0$ b/c it is not defined

Think of $f(x) = \frac{1}{x}$

$\lim_{x \to 0} f(x)$ DNE

LHL $= -\infty$, RHL $= +\infty$

ex $\lim_{x \to 2} \frac{1}{x-2} = ?$



So: $\lim_{x \to 2} \left(\frac{1}{x-2}\right)$ DNE

because LHL $\neq$ RHL

Final Word 2/11

If $\lim_{x \to a} f(x) = c$ and if $f(a) = c$

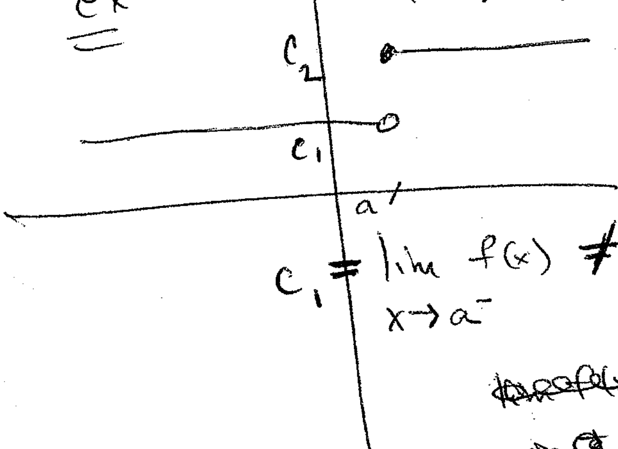
then $f(x)$ is said to be "continuous at $x = c$."

If $f(x)$ is continuous at each x on its domain

then the "fcn. is continuous"

ex

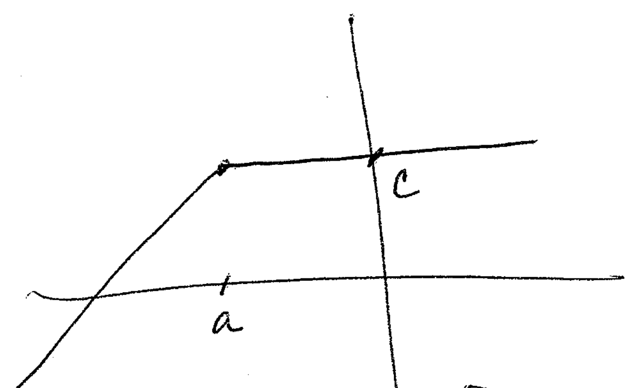
$f(a) = c_2 = \lim_{x \to a^+} f(x)$



$c_1 = \lim_{x \to a^-} f(x) \neq c_2$

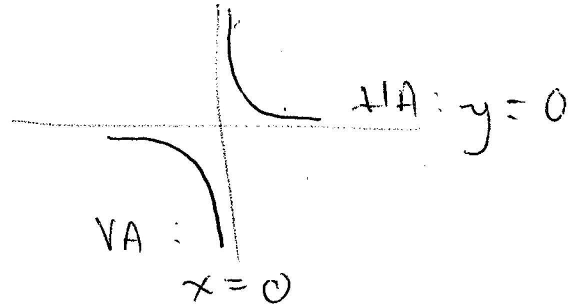
~~$\lim f(x)$~~

So $f(x)$ is not cts. at $x = a$.

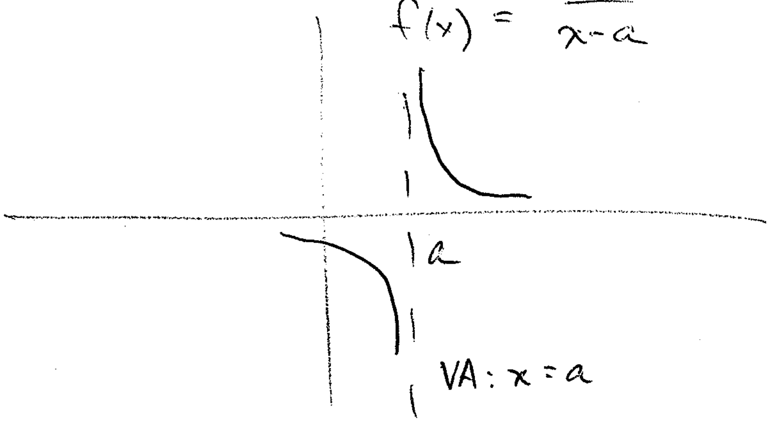


$\left.\begin{array}{l} \lim_{x \to a} f(x) = c \\ \text{and } f(a) = c \end{array}\right\}$ f is cts at $x = a$

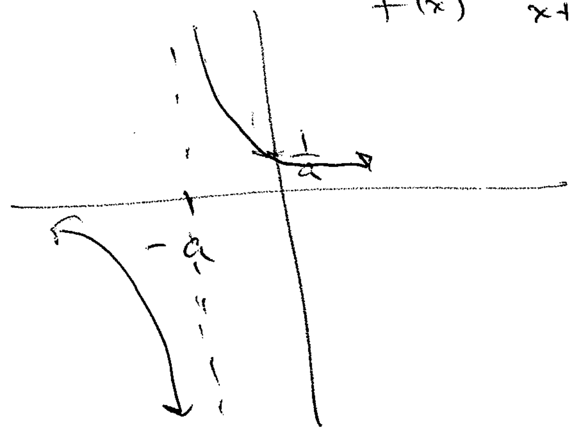
$f(x) = \frac{1}{x}$



$f(x) = \frac{1}{x-a}$

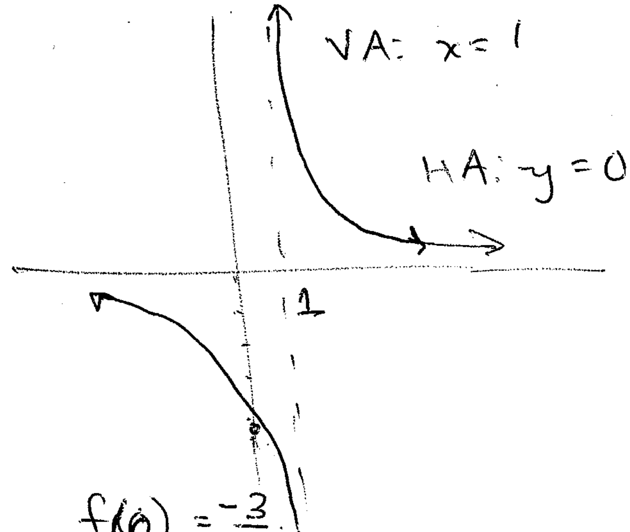


$f(x) = \frac{1}{x+a}$



ex $f(x) = \frac{4}{x-1}$

$f(0) = \frac{4}{0-1} = -4$

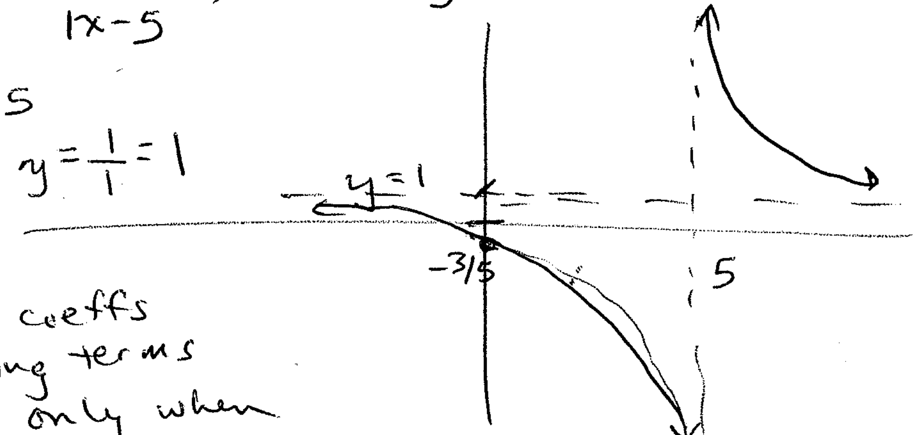


ex $f(x) = \frac{1x+3}{1x-5}$, $f(0) = \frac{-3}{5}$

VA: $x = 5$

HA: ? $y = \frac{1}{1} = 1$

$y = 1$



ratio of the coeffs of the leading terms

- this obtains only when the degree of numerator = degree of denominator