Recall:

$\lim_{x \to a} f(x)$ ≈ formal, mathy "prediction" for the behavior of $f(x)$ @ $x=a$ **based on surrounding pts**

Ex: Consider the following piecewise function

$$f(x) = \begin{cases} x^2+6x+6, & \text{if } x < -2 \\ -2x, & \text{if } -2 \le x \le 2 \\ \frac{x}{2} - 5, & \text{if } x > 2 \end{cases}$$

Evaluate the following

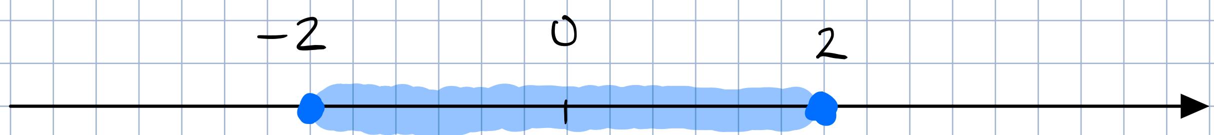
a) $\lim_{x \to 0} f(x)$

the function f spits out various things depending on input

the pt $x=0$ is where we're taking our limit.

Now, for the limit at $x=0$, we're interested only in the **surrounding** x-values (not $x=0$ itself).

Since the x-values surrounding $x=0$ all lie within this interval: $[-2, 2]$



and since $f(x) = -2x$ on this interval, we have

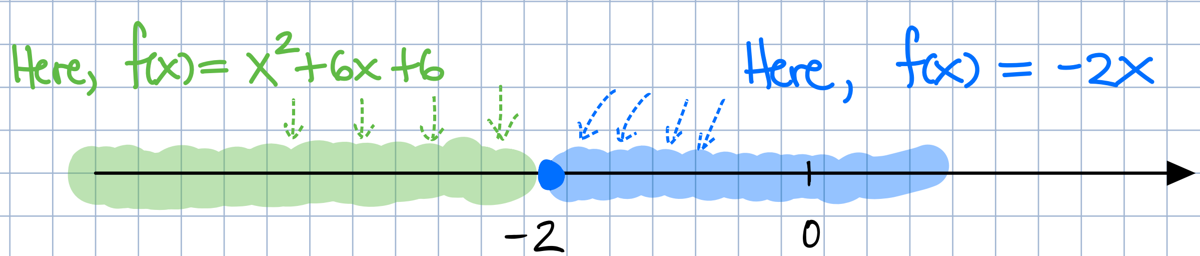
$$\lim_{x \to 0} f(x) = \lim_{x \to 0} -2x = 0$$

b) $\displaystyle\lim_{x \to -2} f(x)$

Just like in part (a), we need to carefully consider which piece we're using to calculate the limit.

Since we're taking the limit @ $x=-2$, we need to investigate those surrounding x-values!

$\Rightarrow$ However, notice that $f(x)$ treats some values diff. than others. It all depends on which interval our x-values reside in.



So, we'll treat the LH & RH separately.

- $\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} x^2+6x+6 = -2$

- $\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} -2x = 4$

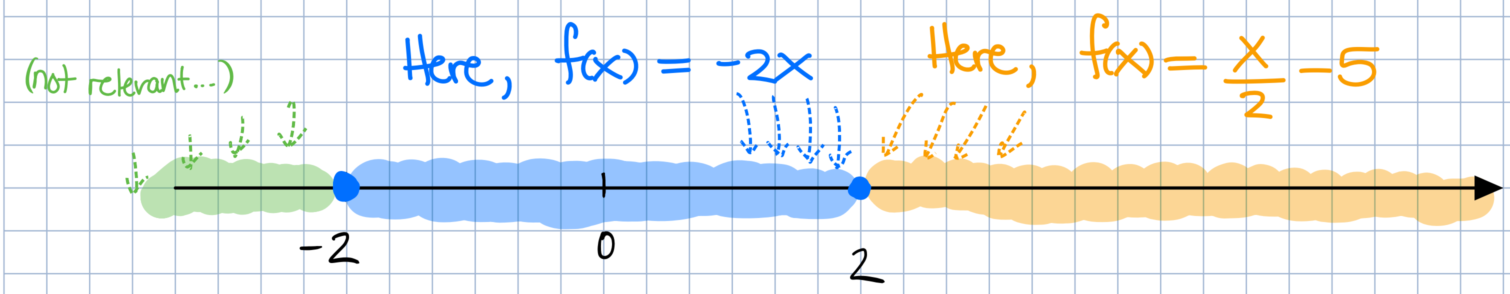
Hence, $\lim_{x \to -2} f(x)$ DNE.

c) $\lim_{x \to 2} f(x)$

This problem is very similar to part b...

Here, we're taking the limit @ $x=2$. So, again we must investigate those surrounding x-values!

$\Rightarrow$ Again, notice that $f(x)$ treats some values diff. than others. It all depends on which interval our x-values reside in.



So,

- $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} -2x = -4$

- $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x}{2} - 5 = -4$

Hence, $\lim_{x \to 2} f(x) = -4$

Infinite Limits: $\rightarrow$ functions that grow without bound

We'll use the following fact often

FACT:

Let f, g be functions defined near $x=a$ so that

① $\lim_{x \to a} f(x) = L$, where $L \neq 0$

② $\lim_{x \to a} g(x) = 0$.

Then, we know that

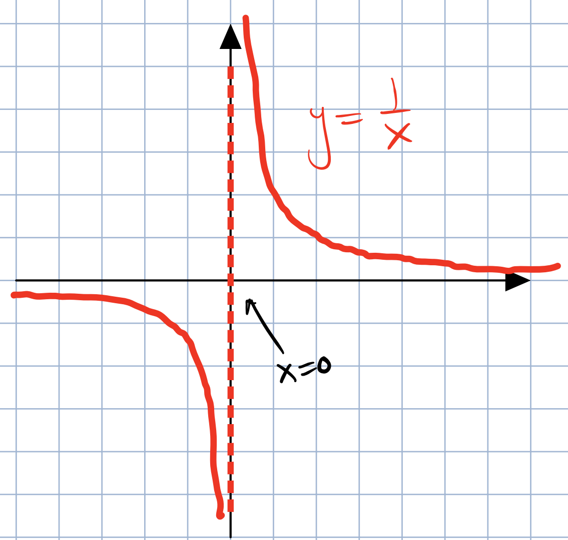
$$\lim_{x \to a^-} \frac{f(x)}{g(x)} = \pm \infty \quad \& \quad \lim_{x \to a^-} \frac{f(x)}{g(x)} = \pm \infty$$

The signs of these limits must be determined by performing additional analysis.

Ex: Consider $f(x) = \frac{1}{x}$.

Find $\lim_{x \to 0} f(x)$.

Before we do any calculations, lets take a look a look at the graph. That way, we'll have an idea of what to expect



Right away, we see that LH/RH behaviors disagree. So, we expect our limit to not exist!

$\Rightarrow$ But! This function is somewhat nice in that although we may not be able to say a limit exists, we can still describe the behaviors of the LH/RH sides.

Notice that our FACT: from earlier still applies

Namely,

$$\lim_{x \to 0} \frac{1}{x} \begin{matrix} \longrightarrow 1 \\ \longrightarrow 0 \end{matrix}$$

So, we know that

- $\lim\limits_{x \to 0^-} \frac{1}{x} = \pm \infty$ & - $\lim\limits_{x \to 0^+} \frac{1}{x} = \pm \infty$

All we have to do now is discover the correct sign.

- $\lim\limits_{x \to 0^-} \frac{1}{x}$ "from the left" $\Rightarrow$ $x < 0$

So $x \to 0$ from the negative side

thus $\lim\limits_{x \to 0^-} \frac{1}{x} \begin{matrix} \longrightarrow 1 \\ \longrightarrow 0^- \end{matrix} \Big\} \; \frac{1}{0^-} \;\; -\infty$ *

*compare with graph

We write $\lim\limits_{x \to 0^-} \frac{1}{x} = -\infty$

(but this technically DNE ; its just a behavior)

Also,

- $\lim_{x \to 0^+} \frac{1}{x}$ "from the right" $\Rightarrow$ $x > 0$

So $x \to 0$ from the positive side

thus $\lim_{x \to 0^+} \frac{1}{x}$ $\begin{matrix} \rightarrow 1 \\ \rightarrow 0^+ \end{matrix}$ $\Big\} +\infty$ *

*compare with graph

We write $\boxed{\lim_{x \to 0^+} \frac{1}{x} = +\infty}$

Ex: Find $\lim_{x \to 2} \frac{3x - 6}{x^2 - 4x + 4}$

1st attempt: plug in if possible.

Here, ... it doesn't work.

2nd attempt: Rewrite with algebra

$$\lim_{x\to 2} \frac{3x-6}{x^2-4x+4} = \lim_{x\to 2} \frac{3\cancel{(x-2)}}{(x-2)\cancel{(x-2)}}$$

$$= \lim_{x\to 2} \frac{3}{x-2} \longrightarrow \frac{3}{0}$$

So, accord. to our fact, the LH-RH limits are infinite:

$$\lim_{x\to 2^-} \frac{3}{x-2} = \pm\infty \quad \& \quad \lim_{x\to 2^+} \frac{3}{x-2} = \pm\infty$$

We just have to figure out which!

So, we proceed as before:

$$\lim_{x\to 2^-} \frac{3}{x-2} \longrightarrow \frac{3}{0^-} \Bigg\} -\infty$$

$$x<2 \Rightarrow (x-2)<0$$

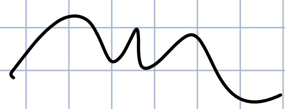
$$\lim_{x\to 2^+} \frac{3}{x-2} \longrightarrow \frac{3}{0^+} \Bigg\} +\infty$$

$$x>2 \Rightarrow (x-2)>0$$

Section 9: Continuous Functions

"continuous" – w/o interruption
no gaps

Intuitively: a cts function is one whose graph has no interruption/gaps.



Def^n (continuous at a pt)

A function f is cts at $a \in D_f$ if

① $\lim_{x \to a} f(x)$ exists,

② $\lim_{x \to a} f(x) = f(a)$

Def^n (continuous on an interval)

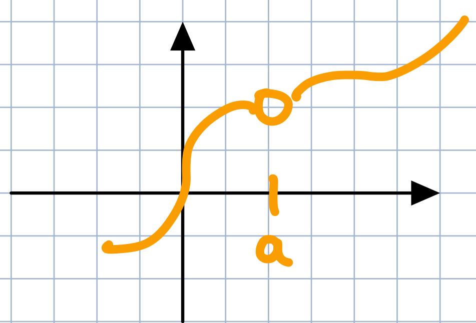
A function f is cts on an interval I if it is cts at every pt in I.

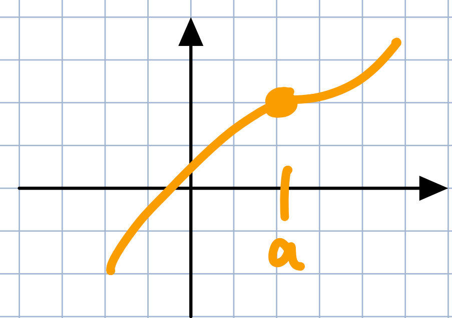
Terminology:

- "f is cts" ⟷ f is cts at every pt in its domain.

- "discontinuous" ⟷ f ain't continuous

Just having a limit vs. Being continuous





Note: the graph of a cts function has no holes

List of Familiar Cts Functions

- **Polynomials** e.g. $p(x) = x^5 - 3x^2 + 2x - 1$

  poly's are cts on $\mathbb{R} = (-\infty, \infty)$

- **Rational Functions** e.g. $\dfrac{2x^2 - 3}{8x^3 + 2x - 1}$

  Domain: anywhere denom $\neq 0$.

  Rational Fcts: are cts on their domain

- **Exponentials / Logarithms**

  → b^x are cts on $(-\infty, \infty)$

  → $\log_b(x)$ are cts on $(0, \infty)$

- **Radicals**

  → $\sqrt[n]{x}$ are cts on their domain.

Ex: Evaluate $\lim_{x \to 1} 3x^2+2x-1$

$f(x) = 3x^2+2x-1$ is cts. So, $\lim_{x \to 1} f(x) = f(1) = 4$