Ex: Suppose you're planning for the future and would like to purchase a \$ 23,000 car four years from now. Your bank offers a savings account offering 2.5% annual interest (compounded monthly). How much should you put away in order to reach your goal?

$F = 23000$, $r = 0.025$

$P = ?$

$n = 12$ (monthly)

$$F = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$23000 = P\left(1 + \frac{.025}{12}\right)^{48}$$

$$\frac{23000}{\left(1+\frac{.025}{12}\right)^{48}} = P \approx \$20,813.43$$

Effects of Compounding:

For what follows, assume that we have a present value P, and an annual interest rate of 3% ($t = 1$ year)

$$F = P\left(1 + \frac{r}{n}\right)^{nt}$$

Let's look at various different n-values:

$n=1$ →	annually	$P(1+.03)$	$= P(1.03)$	$= P(1+.03)$	$= P + P(.03)$
$n=4$ →	quarterly	$P\left(1+\frac{.03}{4}\right)^4$	$= P(1.030339191)$	$= P(1+.030339191)$	$= P + P(.030339191)$
$n=12$ →	monthly	$P\left(1+\frac{.03}{12}\right)^{12}$	$= P(1.030415957)$	$= P(1+.030415957)$	$= P + P(.030415957)$
$n=52$ →	weekly	$P\left(1+\frac{.03}{52}\right)^{52}$	$= P(1.03044562)$	$= P(1+.03044562)$	$= P + P(.03044562)$
$n=365$ →	daily	$P\left(1+\frac{.03}{365}\right)^{365}$	$= P(1.030453264)$	$= P(1+.030453264)$	$= P + P(.030453264)$

We can notice a few things:

- As n (# of compoundings) increases, so too does the "effective interest rate".
- While the effective interest rate increases, it will not increase without bound... it'll eventually "level off."

This convergence is related to a very special constant:

The Number e:

It turns out that as n increases ($n \to \infty$):

$$\boxed{\left(1+\frac{1}{n}\right)^n \approx 2.7182818285\ldots =: e}$$

"Euler's number"

Continuous Compound Interest:

So if $\left(1+\frac{1}{n}\right)^n$ "changes into" e when n is very large, we have 2 questions to answer:

- what effect does this have on our interest formula?
- When is this new formula valid?

If interest is "compounded continuously", this means that n is taken to be very large.

→ Technically, we're assuming that $n \to \infty$ as a limit.

As we saw before, the interest gained (which is determined by the effective interest rate) increases as n increases.

→ For example: if all else is the same other than n, option b is unquestionably more desirable.

OPTION A	**OPTION B**
Present value (P) : \$ 1,000	Present value (P) : \$ 1,000
Annual Interest rate (r) : 2%	Annual Interest rate (r) : 2%
Length of Investment (t) : 1 year	Length of Investment (t) : 1 year
Number of compoundings (n) in a year : $n=2$ (bi-annual)	Number of compoundings (n) in a year : $n = 365$ (daily)

This "continuous compound interest" value represents the best upper bound on the growth of an investment (because of the rapid convergence of $(1+\frac{1}{n})^n \to e$, this limit often coincides with the interest for high n-values, say compounding daily or every hour.)

Finite Compound Interest

Continuous Compound Interest

$$P\left(1+\frac{r}{n}\right)^{nt} \longrightarrow Pe^{rt}$$

FINITE compoundings

INFINITE compoundings

Ex:

Suppose \$600 is invested for 7 years at 4% interest. Find future value.
Find effective interest rate

a) If our bank compounds annually:

$$F = P\left(1+\frac{r}{n}\right)^{nt} \longrightarrow 600\left(1+\frac{.04}{1}\right)^{7}$$

$$= 600(1.31593\ldots)$$

$$= 600(1+.31593\ldots)$$

$$= \underbrace{\$600} + \underbrace{31.59\% \text{ of } \$600}$$

$$= \$789.56$$

(b) If our bank compounds monthly:

$$F = P\left(1+\frac{r}{n}\right)^{nt} \longrightarrow 600\left(1+\frac{0.04}{12}\right)^{84}$$

$$= 600(1.3225139\ldots)$$

$$= 600(1+0.3225139\ldots)$$

$$= \underbrace{\$600} + \underbrace{32.25\% \text{ of } \$600}$$

$$= \$793.51$$

Note: Daily compound interest $(n=365)$ would give \$793.87

(c) If our bank compounds continuously:

$$F = Pe^{rt} \rightarrow 600e^{(0.04)7}$$
$$= 600e^{0.28}$$
$$= 600(1.3231298\ldots)$$
$$= 600(1 + 0.3231298\ldots)$$
$$= \$600 + 32.31\% \text{ on } \$600$$
$$= \$793.88$$

this is the upper bound on compound interest. Meaning:

No matter how many times you compound, you'll never get <u>MORE</u> than this value!

Section 6: Limit

→ Describing the trending behavior of a function at a pt.

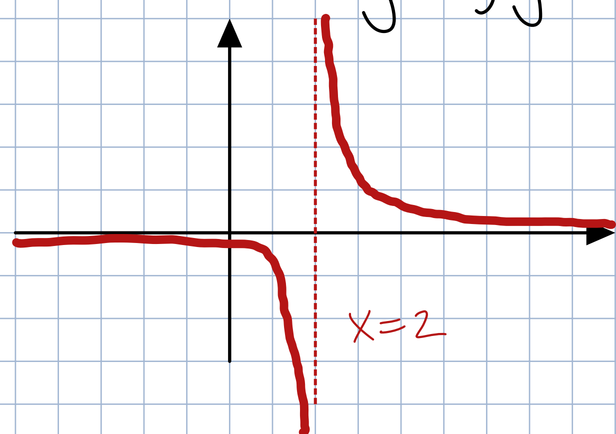
Why?

→ cuz some functions are very ugly

Ex:

$$f(x) = \frac{1}{x-2}$$

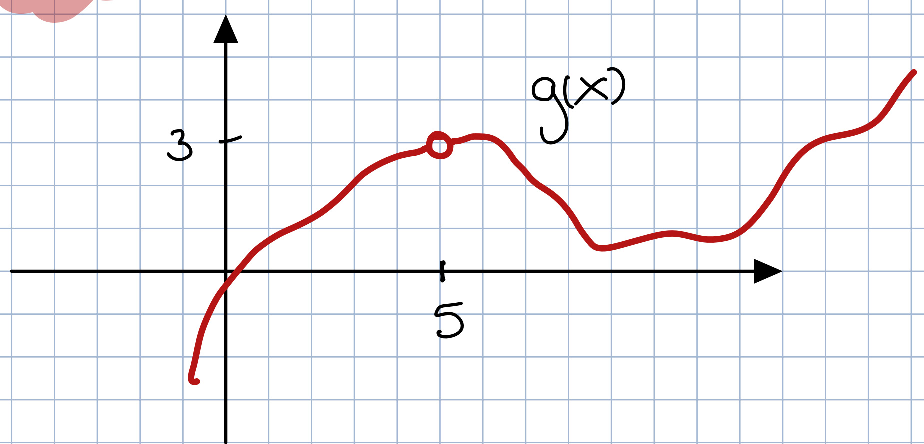
$$D_f = (-\infty, 2) \cup (2, \infty)$$



What does f do at $x=2$?

"$\lim_{x \to 2^-} f(x) = -\infty$" $\longrightarrow$ We'll see how this concept encodes information about f in a precise way.

Ex: Consider the following:



g(5) is und / DNE

Based on the graph, it'd be fair to say that g(x) trends toward 3 as x gets closer to 5.

"$\lim_{x \to 5} g(x) = 3$"

It's helpful to think of a limit as:

Limit = "trending behavior of a fct at a pt".

Def^n

Let f be a function defined near* $x=a$. We say that the limit of $f(x)$ at $x=a$ is $L \in \mathbb{R}$ provided that

- If x is sufficiently close to a, then $f(x)$ can be made arbitrarily close to L.

→ "your outputs $(f(x))$ gotta get close to L whenever x is near a"

<u>Note*:</u> f need not be defined at $x=a$, just on an "open set containing $x=a$"

<u>Notation:</u> If a limit exists, we write

$$\lim_{x \to a} f(x) = L$$

this is read as "the limit of f as x goes to a"

It may be helpful to internally think of the limit notation as

① the y-behavior of

② this function

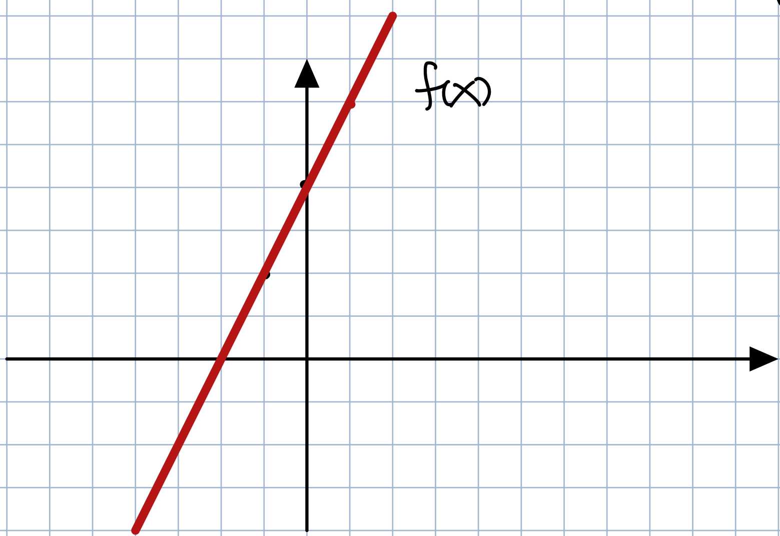
$$\lim_{x \to a} f(x) = L$$

③ based on the x-values near a

Evaluating Limits (1^{st} approach):

Ex: $\lim_{x \to 0} (2x+4)$

→ think of "limit" as "a guess for y-value at x=a" based on surrounding info/values



point surrounding $x=0$:

$\left.\begin{array}{l} -0.02,\ -0.001,\ -0.0003781 \\ 0.03,\ 0.008,\ 0.0000751 \end{array}\right\}$ plug into f(x)

- $f(-0.02) = 2(-0.02) + 4 = 3.96$
- $f(-0.001) = 2(-0.001) + 4 = 3.998$
- $f(0.3) = 2(0.3) + 4 = 4.6$

Based on this, it would be fair to guess that

$\lim_{x \to 0} f(x) = 4$