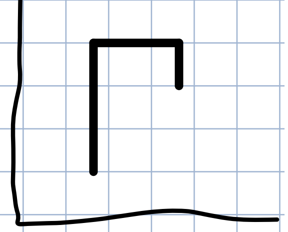
Section 23: Optimization



max/min word problems

Steps:

1) Write down knowns/unknowns
2) Label variables (draw a picture, if applicable)
3) Write down relevant eq^{n}s
    1) Optimizing eq^n
    2) Constraint eq^n
4) Substitute/Differentiate.
5) Use EVT (procedure for abs min/max) or FDT/SDT to find max/min

Example 23.1. If Toys-B-Us charges $p(q)$ cents for a toy, they are able to sell q thousand toys, where $p(q) = 200 - \frac{q}{30}$. How many toys must they sell in order to attain maximum revenue? What is the maximum revenue?

Want: maximize revenue

$q \in [0, \infty)$

Known:

p = price, q = quantity $\Big\}$ $p = 200 - \frac{q}{30}$

constraint eq^n

$\Rightarrow$ Revenue = (price) $\times$ (quantity)

$R(p,q) = p \cdot q$ $\longrightarrow$ optimizing eq^n

$\rightarrow$ Since $p = 200 - \frac{q}{30}$:

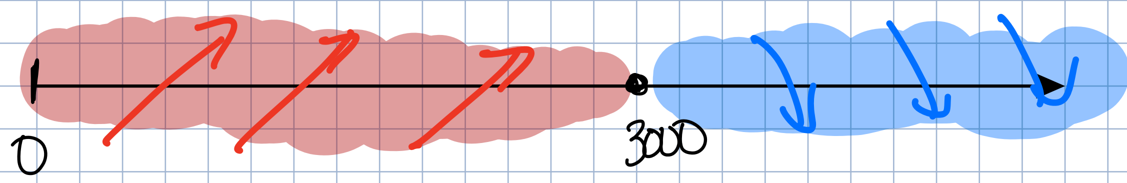
$$R(q) = \left(200 - \frac{q}{30}\right) q = 200q - \frac{q^2}{30}$$

$$\Rightarrow R'(q) = 200 - \frac{q}{15}$$

Crit. Pts: $R'(q) = 0 \Rightarrow q = 3000$

$\Rightarrow$ Verify that $q=3000$ is a max using EVT / FDT / SDT :

FDT:



By FDT, $q=3000$ is a local max.

SDT: $R''(q) = \frac{-1}{15}$ $\rightarrow$ always negative

$R''(3000) = -1/15$ $\rightarrow$ concave down

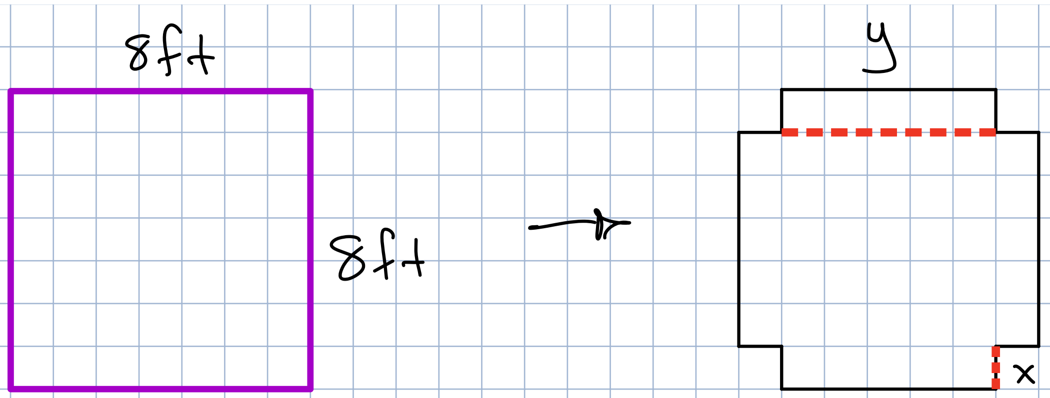
By SDT, $q=3000$ is a local max.

Toys-R-Us sell $q=3000$ toys .

Max Rev: $R(3000) = 200(3000) - \frac{(3000)^2}{30}$

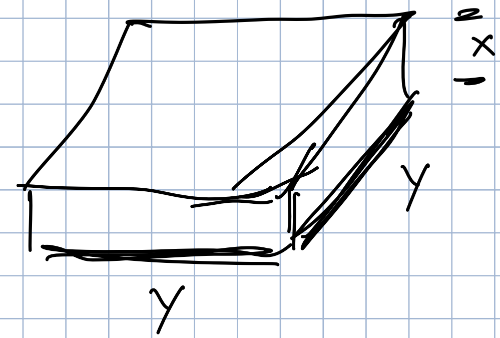
$= \$3000$

Example 23.2. An open box (one without a top) is to be made from an 8 ft. $\times$ 8 ft. square sheet of metal. The box is to be made by cutting out identical squares from each of the four corners of the metal and then bending up the flaps. How large should the cut-out squares be if the box is to have maximal volume? What is the volume of the largest box?



$x \in [0,4]$

Want: maximize volume



Known:

$$\text{Volume} = y^2 x$$

$$V(x,y) = y^2 x$$

Constraint:

$$8 = 2x + y$$

$\Rightarrow$ Use constraint to solve for x or y.

$$\Rightarrow \quad y = 8 - 2x$$

Going back to volume eq^n:

$$V(x, y) = y^2 x$$

$$\Rightarrow \quad V(x) = (8-2x)^2 x$$

$$= (64 - 32x + 4x^2) x$$

$$= 4x^3 - 32x^2 + 64x$$

Find crit. pts and proceed as usual

$$V'(x) = 12x^2 - 64x + 64$$

$$= 4(3x^2 - 16x + 16)$$

$$V'(x) = 0 ?$$

$$x = \frac{16 \pm \sqrt{(16)(16) - 4(3)(16)}}{6}$$

$$= \frac{16 \pm \sqrt{16[16 - 12]}}{6}$$

$$= \frac{16 \pm \sqrt{16(4)}}{6}$$

$$= \frac{16 \pm \sqrt{16}\sqrt{4}}{6} = \frac{16 \pm 8}{6}$$

$x = 4$ $\quad$ $x = 4/3$

SDT: $V''(x) = 4(6x - 16) \Rightarrow x = \frac{8}{3}$

$V''(4) > 0 \Rightarrow$ CU local min

$$V''(4/3) = 4(6(4/3) - 16)$$

$$= -32 < 0 \Rightarrow \text{CD}$$

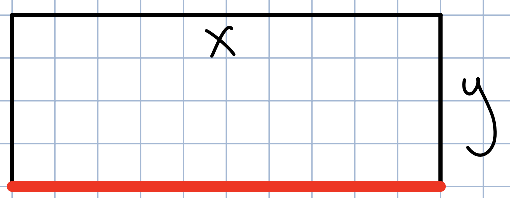
By SDT, $x = 4/3$ is a local max

Volume of largest box:

$$V(4/3) = \left[8 - 2(4/3)\right]^2 (4/3)$$

$$= \left[\frac{16}{3}\right]^2 (4/3) = \frac{1024}{27} \approx 38 \text{ ft}^3$$

Example 23.3. A community service organization has \$6,400 to spend on fencing for a rectangular playground. They want to put fancy fencing on the front and cheaper fencing on the back and sides. Fancy fencing costs \$6 per linear foot. Cheap fencing costs \$2 per linear foot. What are the dimensions of the largest area that can be fenced?



Want: maximize area

Known: $0 \le x \le 800$

opt. eqn $\longrightarrow$ $A(x,y) = x \cdot y$

$$6400 = 2y + 2y + 2x + 6x$$

Constraint. eqn $\to$ $6400 = 4y + 8x$

Solve constraint for x or y:

$$\Rightarrow \quad y = 1600 - 2x$$

So, substituting:

$$A(x) = x(1600 - 2x)$$

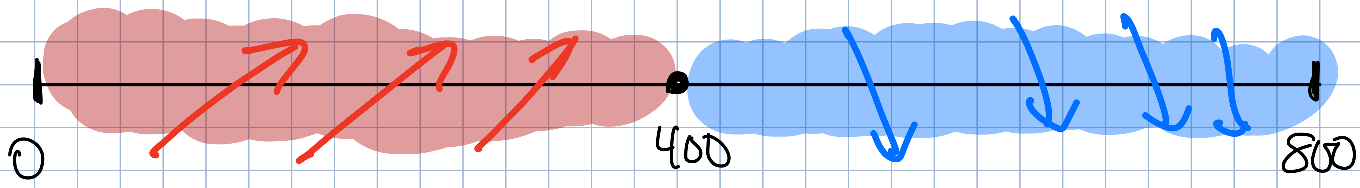
$$A(x) = 1600x - 2x^2$$

Proceed as usual!

$A'(x) = 1600 - 4x \implies$ crit. pt $x = 400$

verify!

FDT: $A'(x) = 1600 - 4x$ $\quad x = 400$



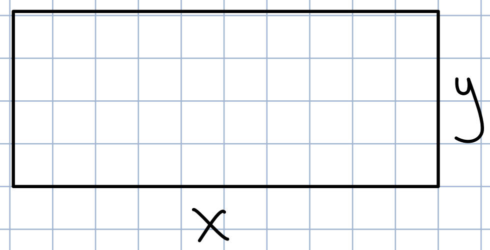
By FDT, $x = 400$ is a local max

To find y: $y = 1600 - 2x$

$$y = 1600 - 2(400) = 800$$

Dimensions: 400 ft $\times$ 800 ft

Example 23.4. Of all rectangles with perimeter 26 cm., what are the dimensions of the one with the largest area?



Want: max area

Known: $A(x,y) = xy$

$$2x + 2y = 26$$

Solve for any variable:

$$x = 13 - y$$

Substituting:

$$A(y) = (13 - y)y$$

$$A(y) = 13y - y^2$$

Proceed as usual!

$$A'(y) = 13 - 2y \implies \text{crit. pts } y = 13/2$$

SDT: $A''(y) = -2 \implies$ always negative
concave down

By SDT, $y = 13/2$ is a local max!

Using $x = 13 - 2y$

$\Rightarrow x = 13/2$

$x = 13/2$ cm

$y = 13/2$ cm