Chapter 5: Polynomials

2/13/2018

$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x^1 + a_0 x^0$

- n^{th} degree polynomial
- a's = coefficients

★ The degree of the polynomial is the highest power of x ★

ex) $f(x) = 4x^5 - 2x^2 + 9$ deg: 5 lead coeff: 4

const: 9 $f(0) = 9$

C-term is always going to be y-int.

ex) $g(x) = 6 - 2x^2 - 4x^4$

deg: 4 leading coeff: -4 (LC always goes w/ 4 deg. term)

const: 6 $g(0) = 6$

ex) $f(x) = (x+1)^3(3x-2)^1 \rightarrow$ not in the correct polynomial form

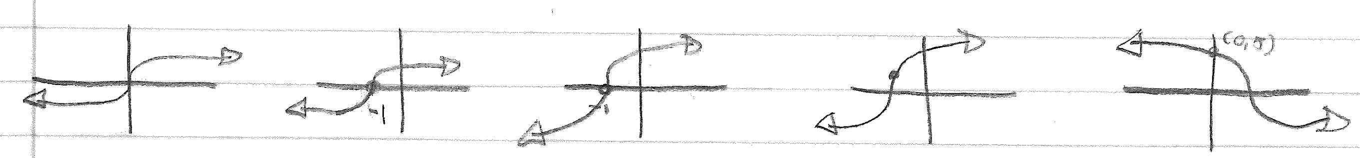
deg: 4 (to find deg. you add exponents of two terms (3+1))

const: -2 (when mult. out -2 is only # w/ no x attached)

lead coeff: 3

2/15/2018

$f(x) = 1 + 4\sqrt[3]{-x+1}$



$y_1 = \sqrt[3]{x}$ $y_2 = \sqrt[3]{x+1}$ $y_3 = 4\sqrt[3]{x+1}$ $y_4 = 1 + 4\sqrt[3]{x+1}$ $y_5 = 1 + 4\sqrt[3]{-x+1}$

$f(0) = 1 + 4\sqrt[3]{-(0)+1}$

$= 1 + 4\sqrt{1}$

$= 1 + 4 \cdot 1$

y-int = 5

$1 + 4\sqrt[3]{-x+1} = 0$

$-1 \qquad -1$

$\frac{4\sqrt[3]{-x+1}}{4} = \frac{-1}{4}$

$(\sqrt[3]{-x+1})^3 = (-\frac{1}{4})^3$

$-x + 1 = \frac{1}{64}$

$-1 \qquad -1$

$\frac{-x}{-1} = \frac{-1\frac{1}{64}}{-1}$

$\boxed{x = 1\frac{1}{64}}$

$$\overset{4}{} \quad + \quad \overset{3}{}$$

$f(x) = (x^{2}+1)^{2}(3-2x)^{3}$

deg: $4+3=7$

LC: $\cancel{1} \cdot (-8) = -8$

Const/Yint: $(0,27)$

$f(x) = -(2-x)^{6}(2x+1)^{4}$

deg: $(1)(6)+(1)(4) = 10$ (look @ x's)

LC: $-(-1)^{6} \cdot 2^{4} = -16$ (look @ x's)

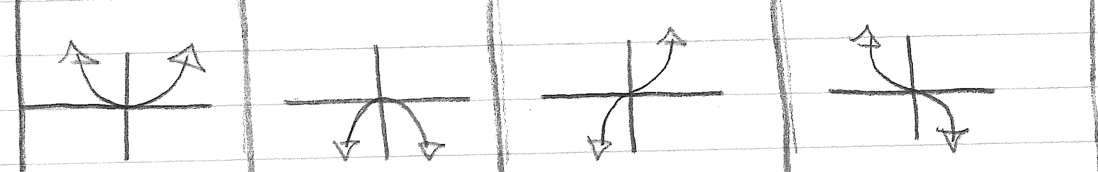
Const/Y-int: $-(2)^{6}(1)^{4} = -64$

End behavior of Polynomials

- As x goes to $+\infty$ or $-\infty$ (4 diff ways)

Memorize Chart ★

① ↑ ↗	② ↙ ↘	③ ↙ ↗	④ ↑ ↓
deg: even lc > 0	deg: even lc < 0	deg: odd lc > 0	deg: odd lc < 0



Some examples...

① $y = x^{2}+x-2$
$y = (x+1)^{2}(x+5)^{2}$

② $y = -x^{4}+2x-8$

③ $y = 5x^{3}+2x+7$

④ $y = -3x^{5}+2x+10$

Second Chart...

lc	> 0	< 0
g en	↑ ↗	↙ ↘
dd	↙ ↗	↑ ↓

Features of the Polynomial to Help us graph It

1. Deg, lc $\rightarrow$ end behavior
2. y-int ($f(0)$)
3. roots $f(x)=0$
4. multiplicity of roots

Ex) $f(x) = (x^2-1)^2(3+2x)^3$

deg: 7 (odd)
lc: $2^3 \cdot 1^2 = 8$

$f(0) = (0^2-1)^2(3+2(0))^3$

$(0, 27)$

X-int $\rightarrow$ $x^2-1=0$

$+1 \quad +1$

$\sqrt{x^2} = \sqrt{1}$

$x = \pm 1$

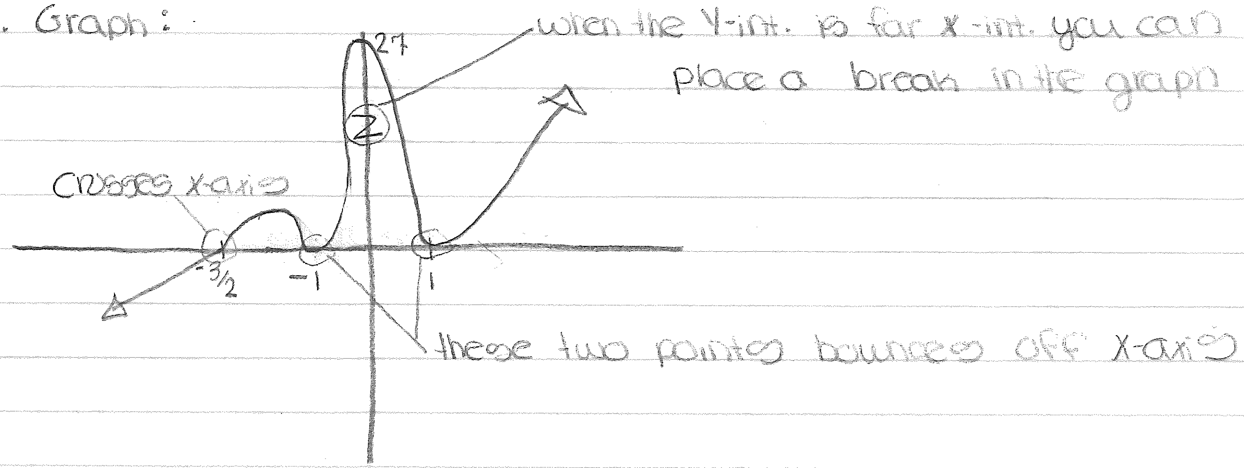
$3+2x=0$

$-3 \quad\quad -3$

$\frac{2x}{2} = \frac{-3}{2}$ $\rightarrow$ $x = -3/2$

Total of 3 roots; these are the points where the polynomial intersects the x-axis

Approx. Graph:



★ Multiplicity of a root is the degree of the factor where root comes from

$(x+1)^2 \quad (x-1)^2 \quad (3+2x)^3$

$x=-1 \quad x=1 \quad x=-3/2$

$M=2 \quad M=2 \quad M=1$

- ☐ If multiplicity is even then line will bounce of x-axis
- ☐ If multiplicity is odd then line will cross through x-axis

$$f(x) = \underbrace{-}_{\text{impacts LC but not deg}}(1+x)^5(x^2-4)$$

deg: $(5)+(2) = 7$ (odd)

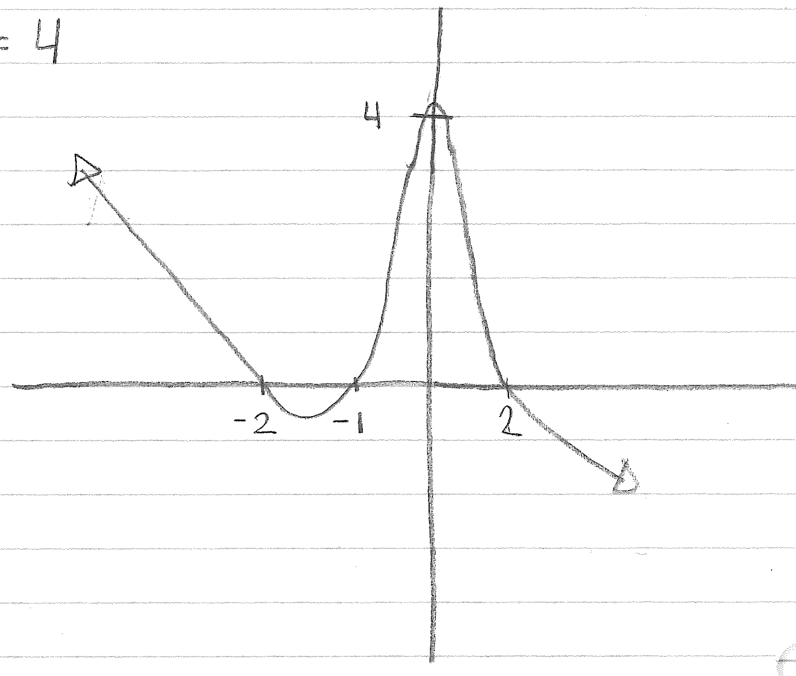
lc: $-(1)^5 \cdot (1)^2 = -1$ (<0)

y-int: $f(0) = -(1+0)^5(0^2-4) = 4$

x-int: $(1+x)(x+2)(x-2)$

$x = -1, 2, -2$

$M = 5$, odd $M = 1$, odd



<u>Lines</u>: 1^{st} degree polynomials

$f(x) = ax+b$ or $a_1x^1 + a_0$

<u>Features of Lines</u>

$f(0) = b$ (y-int is always the constant)

$f(x) = 0$ (@ the one root)

You can have horizontal or slanted lines

Slope-intercept: $y = mx+b$ where slope $= \frac{\text{change in } y}{\text{change in } x}$

To find m choose any two points $(x_1, y_1), (x_2, y_2)$ because for any line the slope is constant

★ Point-slope form: $y - y_1 = m(x - x_1)$

where $m =$ slope, (x_1, y_1) is given point

Graph: $f(x) = -(x+2)^3(x^2-1)^2$ 2/16/2018

deg: $(3) + (2\cdot 2) = 7$ (odd)

lc: $(-1)^3 \cdot (1^2) = -1$ (<0)

y-int: $f(0) = -(0+2)^3(0^2-1)^2$

$= (-8)(1)$

$= -8$

x-int: $(x+2)(x-1)(x+1)$

$x = -2, +1, -1$

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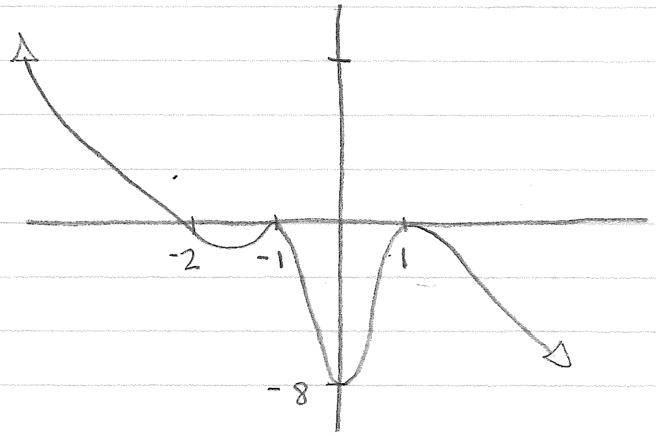
(M=3(odd)) (M=2(even))



$f(x) = x^3 - 27$

$a^3 = x^3$ $b^3 = -27$

$a^2 = x^2$ $b^2 = -9$

$a' = x$ $b' = -3$

$(x-3)(x^2+3x+9) \longrightarrow x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ (roots of Quadratic)

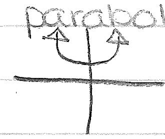
root = 3 | root? - Not Real

$\dfrac{-3 \pm \sqrt{9-4(1)(9)}}{2(1)}$

- when its (-) real roots don't exist

The "floating" parabolas are the ones w/out real roots

Ex: $y = x^2 + 1$



deg: 3

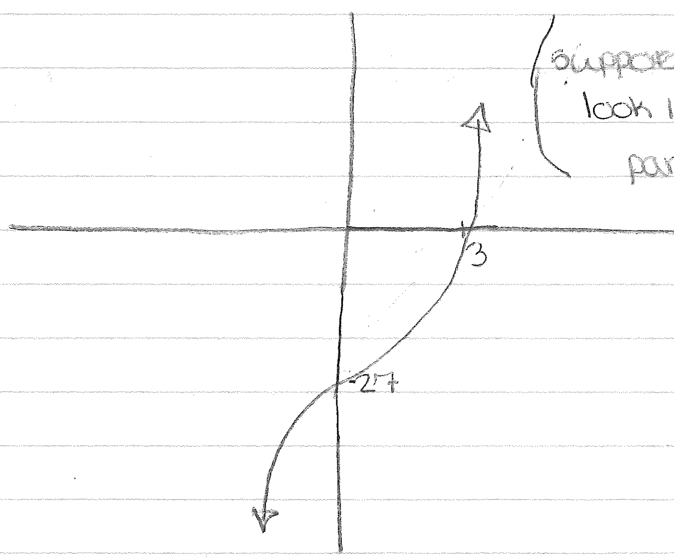
lc: 1 (>0) +

end behavior: ↙ ↗

$f(0) = -27$

root = 3

M=3(odd) - crosses



(supposed to look like x^3 parent function)

Sketch: $(-x^3 - x^2)(+4x + 4)$
deg: 3
lc: < 0 ↑↓
$f(0) = 4$
Roots? - have to factor and solve (using FTA)

Fundamental Theorem of Algebra

- A polynomial $f(x)$ that has a root "r" then $f(r) = 0$ which is equivalent to $x - r$ is a factor
- If "r" is real then it is one of the possible ratios of

$$\left\{ \frac{\text{all factors of constant}}{\text{all factors of lead coeff.}} \right\}$$

→ Write all factors of constant term: 4, -4, 1, -1, 2, -2
Write all factors of lead coeff: 1, -1
Write set of all possible ratios

$$= \left\{ \frac{4}{1}, \frac{-4}{1}, \frac{1}{1}, \frac{-1}{1}, \frac{2}{1}, \frac{-2}{1} \right\} = \{4, -4, 1, -1, 2, -2\}$$

Always check if $x = 1$ is a possible root first

$$f(1) = -1(1^3) - (1^2) + 4(1) + 4$$
$$= -1 - 1 + 8 = 6$$
$$6 \neq 0 \quad (1 \text{ is not a root})$$

$$f(-1) = -1(-1^3) - (1^2) + 4(-1) + 4$$
$$= 1 - 1 - 4 + 4 = 0$$
$$0 = 0 \checkmark \quad -1 \text{ is a root}$$

All possible Real Roots

$$\left\{ \frac{a_0 \text{ factors}}{a_n \text{ factors}} \right\} = \left\{ \pm 1 \text{ is always a factor of everything; therefore always in the set} \right\}$$

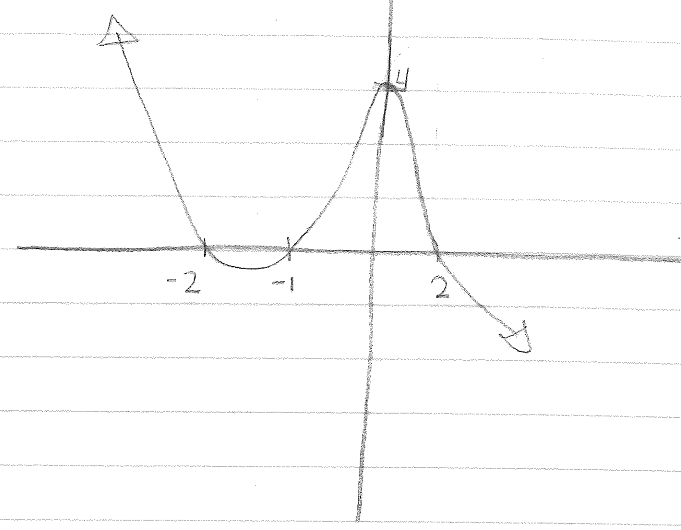
$\boxed{r \text{ is root} \Longleftrightarrow f(r) = 0 \Longleftrightarrow x - r \text{ is a factor}}$

$f(-1) = 0$

$x - (-1) \rightarrow x + 1$ is a factor of the polynomial

To find the other factor(s) divide by $x+1$ aka $(4-x^2)$

$$
\begin{array}{r|l}
 & -x^2 \quad +4 \\
x+1 & -1x^3 - x^2 + 4x + 4 \\
 & \underline{-1x^3 - x^2} \\
 & \qquad\qquad 4x+4 \\
 & \qquad\qquad \underline{4x+4} \\
 & \qquad\qquad\quad 0
\end{array}
$$



Factors: $(x+1)(4-x^2)$

$(x+1)(2-x)(2+x)$

$x = -1, -2, 2$ (roots)

M: all odd

$f(x) = 3x^4 - 2x^2 + x - 6$

Simplified

$\left\{ \pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{3}{3} \pm \frac{6}{3} \right.$

$\left\{ \pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{3}, \pm \frac{2}{3} \pm 6 \right\}$

When you plug in for this one (1, 1) they do not work

Note: We like slope-int. form $y = mx + b$ to graph a line but it isn't helpful if only points are given 2/19/2018

Line Formulas to Know

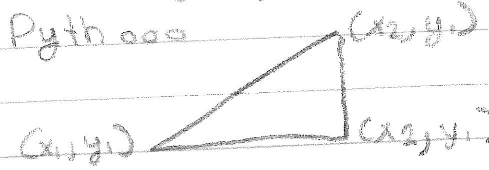
- ☐ Given pt. (x_1, y_1) & slope m the point-slope formula is $y - y_1 = m(x - x_1)$
- ☐ Given two pts. $(x_1, y_1), (x_2, y_2)$, find slope by $m = \frac{y_1 - y_2}{x_1 - x_2}$, then plug m & either pt. into $y - y_1 = m(x - x_1)$

Distance between Points

$(x_1, y_1), (x_2, y_2)$

$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

from Pyth...



Midpoint between 2 Points

$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$

Finally, parallel lines have equal slopes : $m_1 = m_2$

Perpendicular lines' slopes are negative reciprocals of each other

$m_1 = \frac{-1}{m_2}$

find line perpendicular to this one

x) Given 2 pts. $(\overset{x_1}{1}, \overset{y_1}{-2}), (\overset{x_2}{-3}, \overset{y_2}{4})$

Find the equation, distance, and midpoint between them

$m = \frac{4-(-2)}{-3-1} \rightarrow m = \frac{6}{-4} \rightarrow m = -\frac{6}{4} \rightarrow \boxed{\frac{-3}{2}}$

$y-(-2) = (-3/2(x-1) \rightarrow \boxed{y+2 = -3/2(x-1)}$

$D = \sqrt{16 + 36} \rightarrow \sqrt{52} = \boxed{2\sqrt{13}}$

$M = \left(\frac{-3+1}{2}, \frac{4+(-2)}{2} \right) \rightarrow \left(\frac{-2}{2}, \frac{2}{2} \right) \rightarrow \boxed{(-1, 1)}$

$\boxed{y-4 = 2/3(x+3)}$

$-\frac{3}{2} = \frac{-1}{m_2}$ $\quad \frac{-3m_2}{-3} = \frac{-2}{-3}$ $\quad \boxed{m_2 = 2/3}$

2

Synthetic Division 2/19/2018

$x^5 - 5x^4 + 3x^3 + 13x^2 - 8x - 12 \div x + 1$

root $\overline{\big|\text{ coeffs of given}}$

$$\begin{array}{r|rrrrrr} -1 & 1 & -5 & 3 & 13 & -8 & -12 \\ & & -1 & 6 & -9 & -4 & 12 \\ \hline & 1 & -6 & 9 & 4 & -12 & 0 \end{array}$$

(creates a sort of circle)

mult.

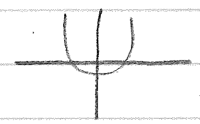
add column

Interpret Row: $1x^4 - 6x^3 + 9x^2 + 4x - 12$

Repeat...

2/20/2018

5.3. Algebraic Approach to the Parabola



$x^2 - 11x = 4$ (final = $\frac{11 \pm \sqrt{137}}{2}$)

$\;\;\;\;\;\;\; -4 \;\;\; -4$

$x^2 - 11x - 4 = 0$

USE: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

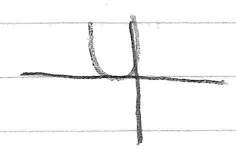
There are two real roots when the # inside the square root is >0 b/c there is a $\pm$

$x = \frac{-(-11) \pm \sqrt{121 - 4(1)(-4)}}{2(1)}$

$b^2 - 4ac$

$121 + 16 = 137 > 0$

two real roots $x^2 - 7 = 0$ (conjugates) $\rightarrow (x - \sqrt{7})(x + \sqrt{7}) = 0$

$y = x^2$  has only 1 root

Quadratics have either 0, 1, or 2 roots

Ex) $x^2 = -3$

$x^2 + 3 = 0$

$X = \frac{0 \pm \sqrt{0-4(1)(3)}}{2} = \frac{\pm \sqrt{-12}}{2}$ ★ No Real Roots b/c of (-) in $\sqrt{\ }$

$b^2 - 4ac$ is known as the "discriminant"

1. $b^2 - 4ac < 0$ (no real roots)
2. $b^2 - 4ac = 0$ (one real root)
3. $b^2 - 4ac > 0$ (two real roots)

Practice

$x^2 + 3x + 5 = 0$ $\quad X = \frac{-(3) \pm \sqrt{(3)^2 - 4(1)(5)}}{2(1)}$

$X = \frac{-3 \pm \sqrt{9-20}}{2}$ ★ no real roots b/c the discriminate is < 0

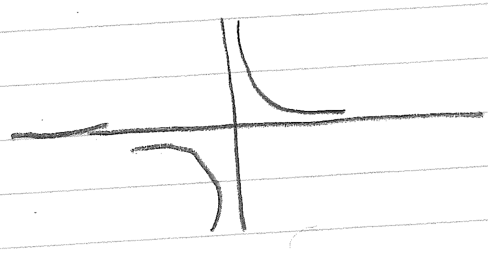
Only parabolas can the quadratic formula be used

Rational Functions are rational expressions set equal to y

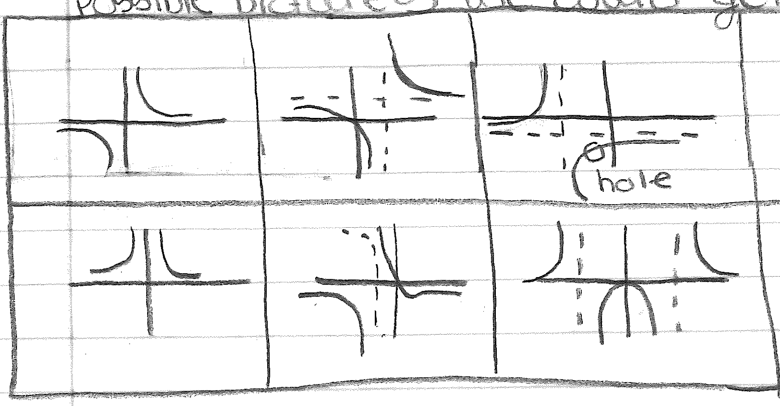
Rational fcns: $f(x) = \frac{P(x)}{Q(x)}$, where P & Q are polynomials

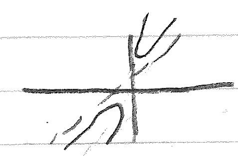
$$= \frac{\boxed{a_n x^n} + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0}{\boxed{b_n x^n} + b_{x-1} x^{n-1} + \cdots}$$

Parent: $f(x) = \frac{1}{x}$



Possible pictures we could get for $f = P/Q$ ...

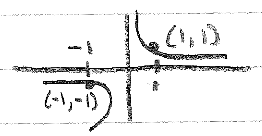




Slant Asymptote (SA) - $y = mx+b$

Be familiar w/ these!

$f(x) = \frac{1}{x}$ → (odd function)



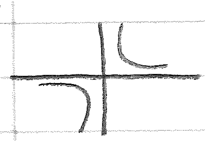
x	$f(x)$
-10	$-1/10$
-2	$-1/2$
$-1/4$	-4
$-1/10$	-10
$x \to 0$	$y \to -\infty$

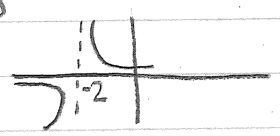
On left $-\infty$

x	$f(x)$
10	$1/10$
2	$1/2$
$1/4$	4
$1/10$	10
$x \to 0$	$y \to \infty$

On right ∞

Ex) $f(x) = \frac{1}{x+2}$

① 

② 

$f(0) = \frac{1}{2}$

Dom: $x \neq -2$

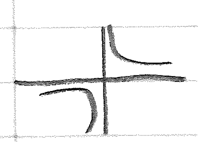
VA: $x = -2$

x-int. → $\frac{1}{x+2} = \frac{0}{1}$ → $1 \neq 0$ no x-intercept

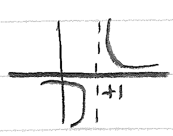
HA: $y = 0$

The VA is the anti-domain (unless it's a hole)

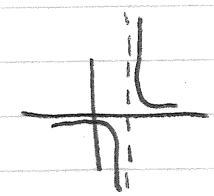
Ex) $f(x) = \frac{3}{x-1} + 4$



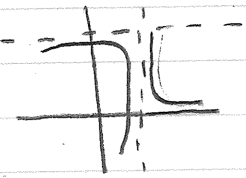
$y_1 = \frac{1}{x}$



$y_2 = \frac{1}{x-1}$



$y_3 = 3y_2$



HA: $y = 4$

VA: $x = 1$

$f(0) = \frac{3}{0-1} + 4$

$\frac{3}{-1} + 4$ yint $= 1$

*hole @ $x = 2$

① $f(x) = \dfrac{x-2}{x^2-4} \to \dfrac{\cancel{(x-2)}}{(x+2)\cancel{(x-2)}} \to \dfrac{1}{x+2}$, where $x \neq 2$

so VA: $x = -2$

Dom: $x \neq \pm 2$

y-int: $f(0) = \dfrac{0-2}{0^2-4} = \dfrac{-2}{-4} = \dfrac{1}{2}$

*The remaining factor in the denominator tells us the VA and The cancelled factor(s) tells us the location of the hole

* VA: usually the "anti-domain" *

② $g(x) = \dfrac{x+1}{x^2-1} \to \dfrac{\cancel{(x+1)}}{\cancel{(x+1)}(x-1)} \to \dfrac{1}{x-1}$, where $x \neq -1$

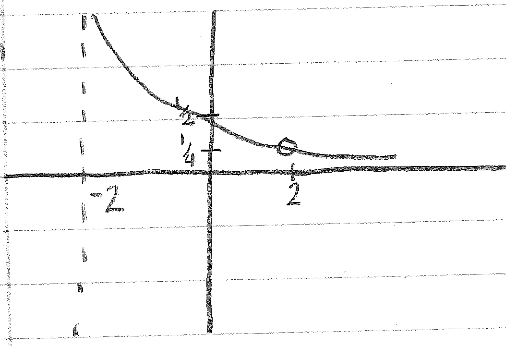
so VA: $x = 1$

hole @ $x = -1$

Dom: $x \neq \pm 1$

y-int: $f(0) = \dfrac{0+1}{(0)^2-1} = \dfrac{1}{-1}$ y-int $= -1$

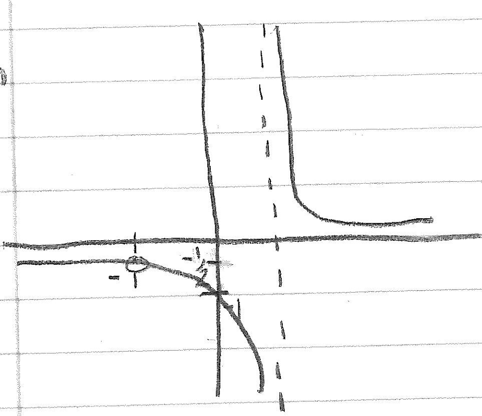
Graph



Hole $(2, f(2)) \to (2, \frac{1}{4})$

$f(2) \downarrow \dfrac{1}{4}$

Graph



$g(-1) = \dfrac{+1}{-1-1} \to -\dfrac{1}{2}$

☐ End Behavior of a rational fcn. is described by the HA and the lead coefficient

$g = \frac{x+1}{x^2-1}$ (from #2 on page before this)

HA: $y=0$

As $x \to +\infty$, $y \to 0$ or $-\infty$

☐ The HA is the guideline for the ends of the fcn as $x \to \pm\infty$

$f(x) = \frac{a_n x^n + \ldots + a_0}{b_m x^m + \ldots + b_m}$

Deg P = n \ HA?; if $n < m$ then $y=0$ is HA

Deg Q = m / why? B/c as $x \to \infty$, Q(x) grows <u>much</u> faster than P(x)

Ex) $f(x) = \frac{\boxed{x^3} + 2x^2 - 4x + 5}{\boxed{x^4} - 1}$ $n < m$ so HA: $y=0$

deg P(x) = 3

deg Q(x) = 4

★ Never name an asymptote with just a number; you have to use an equation

HA: $y = c$

VA: $x = c$

If $n > m$ then there is <u>NO HA</u> because $y \to \infty$ as $x \to \infty$

Ex) $f(x) = \frac{x^3 - 2x + 1}{x+4}$ HA: none (b/c $3 > 1$)

$f(x) = \frac{5}{x+2}$ HA: $y=0$ (bc $0 < 1$)

So far we have $n < m$, then $y = 0$ is HA; $n > m$ then no HA; $n = m$, then $y =$ ratio of the LC $\left(\frac{a_n}{b_m}\right)$ is HA

Ex) $f(x) = \dfrac{9x^3 + 2x - 3}{4x^3 + 7x}$

HA: $y = \dfrac{9}{4}$