

Slope and Linear Functions

Horizontal and Vertical Lines

Let's consider graphs of equations $y = c$ and $x = a$, where c and a are real numbers.

■ EXAMPLE 1

- a) Graph $y = 4$.
- b) Decide whether the graph represents a function.

Solution

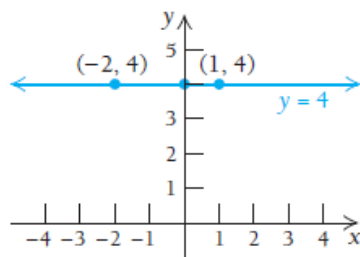
- a) The graph consists of all ordered pairs whose second coordinate is 4. To see how a pair such as $(-2, 4)$ could be a solution of $y = 4$, we can consider the equation above in the form

$$y = 0x + 4.$$

Then $(-2, 4)$ is a solution because

$$0(-2) + 4 = 4$$

is true.



- b) The vertical-line test holds. Thus, the graph represents a function.
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■ EXAMPLE 2

- a) Graph $x = -3$.
- b) Decide whether the graph represents a function.

Solution

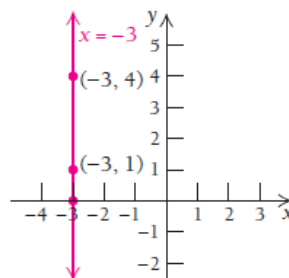
- a) The graph consists of all ordered pairs whose first coordinate is -3 . To see how a pair such as $(-3, 4)$ could be a solution of $x = -3$, we can consider the equation in the form

$$x + 0y = -3.$$

Then $(-3, 4)$ is a solution because

$$(-3) + 0(4) = -3$$

is true.



- b) This graph does not represent a function because it fails the vertical-line test. The line itself meets the graph more than once—in fact, infinitely many times.

◀ Quick Check 1

In general, we have the following.

THEOREM 3

The graph of $y = c$, or $f(x) = c$, a horizontal line, is the graph of a function. Such a function is referred to as a constant function. The graph of $x = a$ is a vertical line, and $x = a$ is not a function.

The Equation $y = mx$

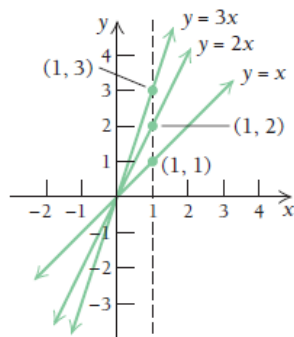
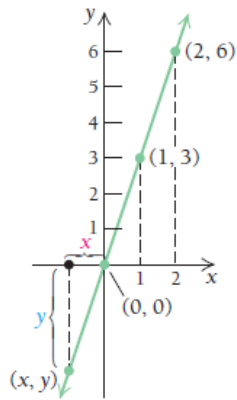
Consider the following table of numbers and look for a pattern.

x	1	-1	$-\frac{1}{2}$	2	-2	3	-7	5
y	3	-3	$-\frac{3}{2}$	6	-6	9	-21	15

Note that the ratio of the y -value to the x -value is 3. That is,

$$\frac{y}{x} = 3, \quad \text{or} \quad y = 3x.$$

Ordered pairs from the table can be used to graph the equation $y = 3x$ (see the figure at the left). Note that this is a function.



THEOREM 4

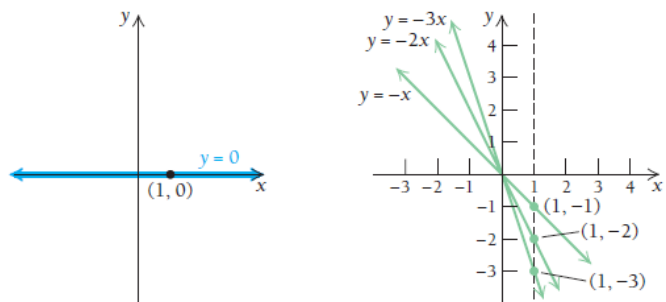
The graph of the function given by

$$y = mx \quad \text{or} \quad f(x) = mx$$

is the straight line through the origin $(0, 0)$ and the point $(1, m)$. The constant m is called the slope of the line.

Various graphs of $y = mx$ for positive values of m are shown to the left. Note that such graphs slant up from left to right. A line with large positive slope rises faster than a line with smaller positive slope.

When $m = 0$, $y = 0x$, or $y = 0$. On the left below is a graph of $y = 0$. Note that this is both the x -axis and a horizontal line.



Graphs of $y = mx$ for negative values of m are shown on the right above. Note that such graphs slant down from left to right.

Direct Variation

There are many applications involving equations like $y = mx$, where m is some positive number. In such situations, we say that we have direct variation, and m (the slope) is called the variation constant, or constant of proportionality. Generally, only positive values of x and y are considered.

DEFINITION

The variable y varies directly as x if there is some positive constant m such that $y = mx$. We also say that y is directly proportional to x .

■ **EXAMPLE 3** Life Science: Weight on Earth and the Moon. The weight M , in pounds, of an object on the moon is directly proportional to the weight E of that object on Earth. An astronaut who weighs 180 lb on Earth will weigh 28.8 lb on the moon.

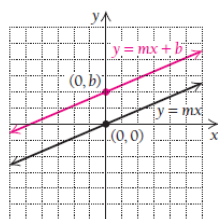
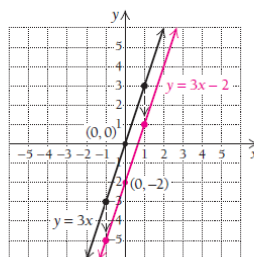
- a) Find an equation of variation.
- b) An astronaut weighs 120 lb on Earth. How much will the astronaut weigh on the moon?

The Equation $y = mx + b$

Compare the graphs of the equations

$$y = 3x \quad \text{and} \quad y = 3x - 2$$

(see the following figure). Note that the graph of $y = 3x - 2$ is a shift 2 units down of the graph of $y = 3x$, and that $y = 3x - 2$ has y -intercept $(0, -2)$. Both graphs represent functions.



DEFINITION

A linear function is given by

$$y = mx + b \quad \text{or} \quad f(x) = mx + b$$

and has a graph that is the straight line parallel to the graph of $y = mx$ and crossing the y -axis at $(0, b)$. The point $(0, b)$ is called the y -intercept. (See the figure at the left.)

As before, the constant m is the slope of the line. When $m = 0$, $y = 0x + b = b$, and we have a constant function (see Theorem 3 at the beginning of this section). The graph of such a function is a horizontal line.

The Slope–Intercept Equation

Every nonvertical line l is uniquely determined by its slope m and its y -intercept $(0, b)$. In other words, the slope describes the “slant” of the line, and the y -intercept locates the point at which the line crosses the y -axis. Thus, we have the following definition.

DEFINITION

$y = mx + b$ is called the slope–intercept equation of a line.

■ **EXAMPLE 4** Find the slope and the y -intercept of the graph of $2x - 4y - 7 = 0$.

Solution We solve for y :

$$2x - 4y - 7 = 0$$

$$4y = 2x - 7 \quad \text{Adding } 4y \text{ to both sides}$$

$$y = \frac{2}{4}x - \frac{7}{4} \quad \text{Dividing both sides by 4}$$

$$\text{Slope: } \frac{1}{2} \quad \text{y-intercept: } \left(0, -\frac{7}{4}\right)$$

The Point–Slope Equation

Suppose that we know the slope of a line and some point on the line other than the y-intercept. We can still find an equation of the line.

■ **EXAMPLE 5** Find an equation of the line with slope 3 containing the point $(-1, -5)$.

Solution The slope is given as $m = 3$. From the slope–intercept equation, we have

$$y = 3x + b, \quad (1)$$

so we must determine b . Since $(-1, -5)$ is on the line, we substitute -5 for y and -1 for x :

$$-5 = 3(-1) + b$$

$$-5 = -3 + b,$$

so
$$-2 = b$$

Then, replacing b in equation (1) with -2 , we get $y = 3x - 2$.

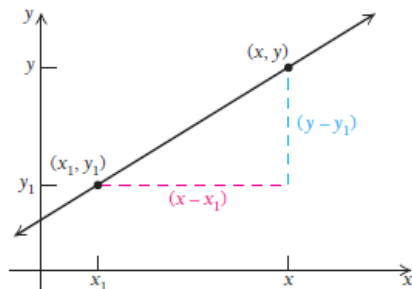
More generally, if a point (x_1, y_1) is on the line given by

$$y = mx + b, \quad (2)$$

it must follow that

$$y_1 = mx_1 + b. \quad (3)$$

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Subtracting the left and right sides of equation (3) from the left and right sides, respectively, of equation (2), we have

$$\begin{aligned}y - y_1 &= (mx + b) - (mx_1 + b) \\&= mx + b - mx_1 - b && \text{Multiplying by } -1 \\&= mx - mx_1 && \text{Combining like terms} \\&= m(x - x_1). && \text{Factoring}\end{aligned}$$

DEFINITION

$y - y_1 = m(x - x_1)$ is called the **point-slope equation** of a line. The point is (x_1, y_1) , and the slope is m .

This definition allows us to write an equation of a line given its slope and the coordinates of *any* point on the line.

EXAMPLE 6 Find an equation of the line with slope $\frac{2}{3}$ containing the point $(-1, -5)$.

Solution Substituting in

$$y - y_1 = m(x - x_1),$$

we get

$$y - (-5) = \frac{2}{3}[x - (-1)]$$

$$y + 5 = \frac{2}{3}(x + 1)$$

$$y + 5 = \frac{2}{3}x + \frac{2}{3} \quad \text{Multiplying by } \frac{2}{3}$$

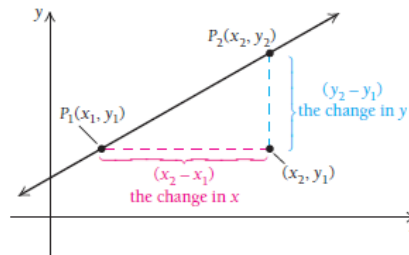
$$y = \frac{2}{3}x + \frac{2}{3} - 5 \quad \text{Subtracting 5}$$

$$y = \frac{2}{3}x + \frac{2}{3} - \frac{15}{3}$$

$$y = \frac{2}{3}x - \frac{13}{3}. \quad \text{Combining like terms}$$

Computing Slope

We now determine a method of computing the slope of a line when we know the coordinates of two of its points. Suppose that (x_1, y_1) and (x_2, y_2) are the coordinates of two different points, P_1 and P_2 , respectively, on a line that is not vertical. Consider a right triangle with legs parallel to the axes, as shown in the following figure.



Note that the change in y is $y_2 - y_1$ and the change in x is $x_2 - x_1$. The ratio of these changes is the slope. To see this, consider the point-slope equation,

$$y - y_1 = m(x - x_1).$$

Since (x_2, y_2) is on the line, it must follow that

$$y_2 - y_1 = m(x_2 - x_1). \quad \text{Substituting}$$

Since the line is not vertical, the two x -coordinates must be different; thus, $x_2 - x_1$ is nonzero, and we can divide by it to get the following theorem.

THEOREM 5

The slope of a line containing points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x}.$$

■ **EXAMPLE 7** Find the slope of the line containing the points $(-2, 6)$ and $(-4, 9)$.

Solution We have

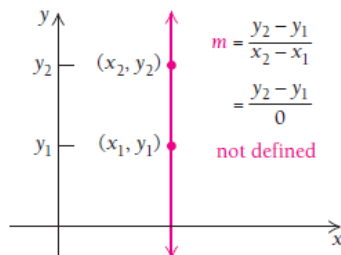
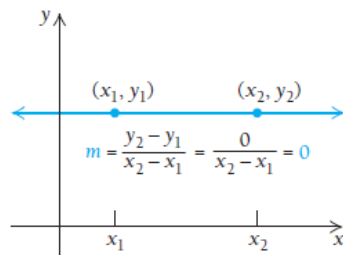
$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 9}{-2 - (-4)} && \text{We treated } (-2, 6) \text{ as } P_2 \text{ and } (-4, 9) \text{ as } P_1. \\ &= \frac{-3}{2} = -\frac{3}{2}. \end{aligned}$$

Note that it does not matter which point is taken first, so long as we subtract the coordinates in the same order. In this example, we can also find m as follows:

$$m = \frac{9 - 6}{-4 - (-2)} = \frac{3}{-2} = -\frac{3}{2}. \quad \text{Here, } (-4, 9) \text{ serves as } P_2, \text{ and } (-2, 6) \text{ serves as } P_1.$$

◀ **Quick Check 5**

If a line is horizontal, the change in y for any two points is 0. Thus, a horizontal line has slope 0. If a line is vertical, the change in x for any two points is 0. Thus, the slope is *not defined* because we cannot divide by 0. A vertical line has undefined slope. Thus, “0 slope” and “undefined slope” are two very different concepts.



Applications of Linear Functions

Many applications are modeled by linear functions.

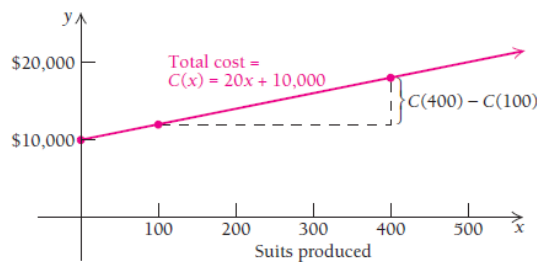
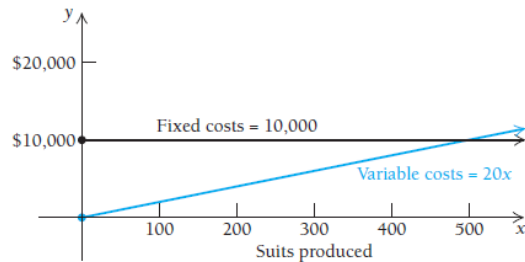
EXAMPLE 9 Business: Total Cost. Raggs, Ltd., a clothing firm, has fixed costs of \$10,000 per year. These costs, such as rent, maintenance, and so on, must be paid no matter how much the company produces. To produce x units of a certain kind of suit, it costs \$20 per suit (unit) in addition to the fixed costs. That is, the variable costs for producing x of these suits are $20x$ dollars. These costs are due to the amount produced and stem from items such as material, wages, fuel, and so on. The total cost $C(x)$ of producing x suits in a year is given by a function C :

$$C(x) = (\text{Variable costs}) + (\text{Fixed costs}) = 20x + 10,000.$$

- Graph the variable-cost, the fixed-cost, and the total-cost functions.
- What is the total cost of producing 100 suits? 400 suits?

Solution

- The variable-cost and fixed-cost functions appear in the graph on the left below. The total-cost function is shown in the graph on the right. From a practical standpoint, the domains of these functions are nonnegative integers 0, 1, 2, 3, and so on, since it does not make sense to make either a negative number or a fractional number of suits. It is common practice to draw the graphs as though the domains were the entire set of nonnegative real numbers.



- b) The total cost of producing 100 suits is

$$C(100) = 20 \cdot 100 + 10,000 = \$12,000.$$

The total cost of producing 400 suits is

$$\begin{aligned} C(400) &= 20 \cdot 400 + 10,000 \\ &= \$18,000. \end{aligned}$$

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■ **EXAMPLE 10** Business: Profit-and-Loss Analysis. When a business sells an item, it receives the *price* paid by the consumer (this is normally greater than the *cost* to the business of producing the item).

- a) The total revenue that a business receives is the product of the number of items sold and the price paid per item. Thus, if Raggs, Ltd., sells x suits at \$80 per suit, the total revenue $R(x)$, in dollars, is given by

$$R(x) = \text{Unit price} \cdot \text{Quantity sold} = 80x.$$

If $C(x) = 20x + 10,000$ (see Example 9), graph R and C using the same set of axes.

- b) The total profit that a business receives is the amount left after all costs have been subtracted from the total revenue. Thus, if $P(x)$ represents the total profit when x items are produced and sold, we have

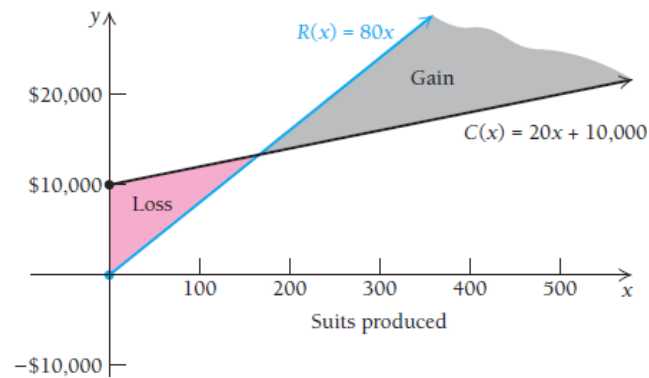
$$P(x) = (\text{Total revenue}) - (\text{Total costs}) = R(x) - C(x).$$

Determine $P(x)$ and draw its graph using the same set of axes as was used for the graph in part (a).

- c) The company will *break even* at that value of x for which $P(x) = 0$ (that is, no profit and no loss). This is the point at which $R(x) = C(x)$. Find the break-even value of x .

Solution

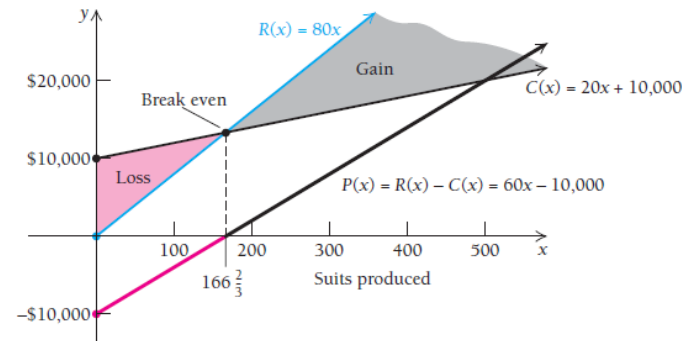
- a) The graphs of $R(x) = 80x$ and $C(x) = 20x + 10,000$ are shown below. When $C(x)$ is above $R(x)$, a loss will occur. This is shown by the region shaded red. When $R(x)$ is above $C(x)$, a gain will occur. This is shown by the region shaded gray.



b) To find P , the profit function, we have

$$\begin{aligned} P(x) &= R(x) - C(x) = 80x - (20x + 10,000) \\ &= 60x - 10,000. \end{aligned}$$

The graph of $P(x)$ is shown by the heavy line. The red portion of the line shows a “negative” profit, or loss. The black portion of the heavy line shows a “positive” profit, or gain.



c) To find the break-even value, we solve $R(x) = C(x)$:

$$\begin{aligned} R(x) &= C(x) \\ 80x &= 20x + 10,000 \\ 60x &= 10,000 \\ x &= 166 \frac{2}{3}. \end{aligned}$$

How do we interpret the fractional answer, since it is not possible to produce $\frac{2}{3}$ of a suit? We simply round up to 167. Estimates of break-even values are usually sufficient since companies want to operate well away from break-even values in order to maximize profit.

Quick Check 6

Business. Suppose that in Examples 9 and 10 fixed costs are increased to \$20,000. Find:

- the total-cost, total-revenue, and total-profit functions;
- the break-even value.

Quick Check 6

Graph.

1. $x = 3$
2. $x = 5$
3. $y = -2$
4. $y = -4$
5. $x = -4.5$
6. $x = -1.5$
7. $y = 3.75$
8. $y = 2.25$

Graph. List the slope and y-intercept.

9. $y = -2x$
10. $y = -3x$
11. $f(x) = 0.5x$
12. $f(x) = -0.5x$
13. $y = 3x - 4$
14. $y = 2x - 5$
15. $g(x) = -x + 3$
16. $g(x) = x - 2.5$
17. $y = 7$
18. $y = -5$

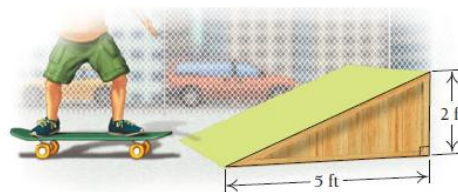
Find the slope and y-intercept.

19. $y - 3x = 6$
20. $y - 4x = 1$
21. $2x + y - 3 = 0$
22. $2x - y + 3 = 0$
23. $2x + 2y + 8 = 0$
24. $3x - 3y + 6 = 0$
25. $x = 3y + 7$
26. $x = -4y + 3$

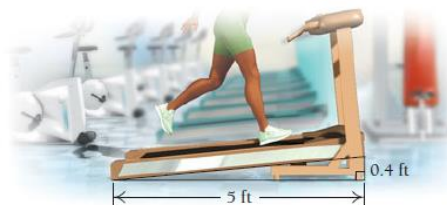
Find an equation of the line:

27. with $m = -5$, containing $(-2, -3)$.
28. with $m = 7$, containing $(1, 7)$.
29. with $m = -2$, containing $(2, 3)$.
30. with $m = -3$, containing $(5, -2)$.
31. with slope 2, containing $(3, 0)$.
32. with slope -5 , containing $(5, 0)$.
33. with y-intercept $(0, -6)$ and slope $\frac{1}{2}$.
34. with y-intercept $(0, 7)$ and slope $\frac{4}{3}$.
35. with slope 0, containing $(2, 3)$.

39. $(2, -3)$ and $(-1, -4)$
40. $(-3, -5)$ and $(1, -6)$
41. $(3, -7)$ and $(3, -9)$
42. $(-4, 2)$ and $(-4, 10)$
43. $(\frac{4}{5}, -3)$ and $(\frac{1}{2}, \frac{2}{5})$
44. $(-\frac{3}{16}, -\frac{1}{2})$ and $(\frac{2}{8}, -\frac{3}{4})$
45. $(2, 3)$ and $(-1, 3)$
46. $(-6, \frac{1}{2})$ and $(-7, \frac{1}{2})$
47. $(x, 3x)$ and $(x + h, 3(x + h))$
48. $(x, 4x)$ and $(x + h, 4(x + h))$
49. $(x, 2x + 3)$ and $(x + h, 2(x + h) + 3)$
50. $(x, 3x - 1)$ and $(x + h, 3(x + h) - 1)$
- 51–60. Find an equation of the line containing the pair of points in each of Exercises 37–46.
61. Find the slope of the skateboard ramp.



62. Find the slope (or grade) of the treadmill.



63. Find the slope (or head) of the river. Express the answer as a percentage.



APPLICATIONS

Business and Economics

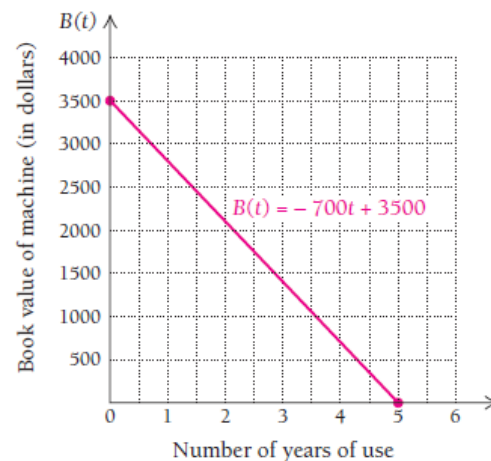
64. **Highway tolls.** It has been suggested that since heavier vehicles are responsible for more of the wear and tear on highways, drivers should pay tolls in direct proportion to the weight of their vehicles. Suppose that a Toyota Camry weighing 3350 lb was charged \$2.70 for traveling an 80-mile stretch of highway.
- Find an equation of variation that expresses the amount of the toll T as a function of the vehicle's weight w .
 - What would the toll be if a 3700-lb Jeep Cherokee drove the same stretch of highway?
65. **Inkjet cartridges.** A registrar's office finds that the number of inkjet cartridges, I , required each year for its copiers and printers varies directly with the number of students enrolled, s .
- Find an equation of variation that expresses I as a function of s , if the office requires 16 cartridges when 2800 students enroll.
 - How many cartridges would be required if 3100 students enrolled?
66. **Profit-and-loss analysis.** Boxowitz, Inc., a computer firm, is planning to sell a new graphing calculator. For the first year, the fixed costs for setting up the new production line are \$100,000. The variable costs for producing each calculator are estimated at \$20. The sales department projects that 150,000 calculators can be sold during the first year at a price of \$45 each.
- Find and graph $C(x)$, the total cost of producing x calculators.
 - Using the same axes as in part (a), find and graph $R(x)$, the total revenue from the sale of x calculators.
 - Using the same axes as in part (a), find and graph $P(x)$, the total profit from the production and sale of x calculators.
 - What profit or loss will the firm realize if the expected sale of 150,000 calculators occurs?
68. **Straight-line depreciation.** Quick Copy buys an office machine for \$5200 on January 1 of a given year. The machine is expected to last for 8 yr, at the end of which time its *salvage value* will be \$1100. If the company figures the decline in value to be the same each year, then the *book value*, $V(t)$, after t years, $0 \leq t \leq 8$, is given by
- $$V(t) = C - t\left(\frac{C - S}{N}\right),$$
- where C is the original cost of the item, N is the number of years of expected life, and S is the salvage value.
- Find the linear function for the straight-line depreciation of the office machine.
 - Find the book value after 0 yr, 1 yr, 2 yr, 3 yr, 4 yr, 7 yr, and 8 yr.
69. **Profit-and-loss analysis.** Jimmy decides to mow lawns to earn money. The initial cost of his lawnmower is \$250. Gasoline and maintenance costs are \$4 per lawn.
- Formulate a function $C(x)$ for the total cost of mowing x lawns.
 - Jimmy determines that the total-profit function for the lawnmowing business is given by $P(x) = 9x - 250$. Find a function for the total revenue from mowing x lawns. How much does Jimmy charge per lawn?
 - How many lawns must Jimmy mow before he begins making a profit?
70. **Straight-line depreciation.** (See Exercise 68.) A business tenant spends \$40 per square foot on improvements to a 25,000-ft² office space. Under IRS guidelines for straight-line depreciation, these improvements will depreciate completely—that is, have zero salvage value—after 39 yr. Find the depreciated value of the improvements after 10 yr.
71. **Book value.** (See Exercise 68.) The Video Wizard buys a new computer system for \$60,000 and projects that its book value will be \$2000 after 5 yr. Using straight-line depreciation, find the book value after 3 yr.
72. **Book value.** Tyline Electric uses the function $B(t) = -700t + 3500$ to find the book value, $B(t)$, in dollars, of a photocopier t years after its purchase.

- e) How many calculators must the firm sell in order to break even?

67. **Profit-and-loss analysis.** Red Tide is planning a new line of skis. For the first year, the fixed costs for setting up production are \$45,000. The variable costs for producing each pair of skis are estimated at \$80, and the selling price will be \$255 per pair. It is projected that 3000 pairs will sell the first year.

- Find and graph $C(x)$, the total cost of producing x pairs of skis.
- Find and graph $R(x)$, the total revenue from the sale of x pairs of skis. Use the same axes as in part (a).
- Using the same axes as in part (a), find and graph $P(x)$, the total profit from the production and sale of x pairs of skis.
- What profit or loss will the company realize if the expected sale of 3000 pairs occurs?
- How many pairs must the company sell in order to break even?

dollars, of a photocopier t years after its purchase.

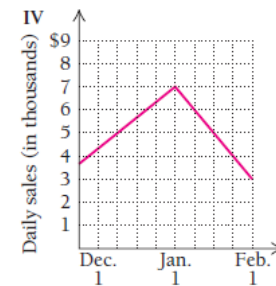
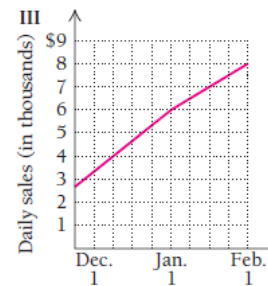


- What do the numbers -700 and 3500 signify?
- How long will it take the copier to depreciate completely?
- What is the domain of B ? Explain.

- b) Use the equation in part (a) to estimate the number of manatees counted in January 2010.
- c) The actual number counted in January 2010 was 5067. Does the equation found in part (a) give an accurate representation of the number of manatees counted each year?
87. **Urban population.** The population of Woodland is P . After a growth of 2%, its new population is N .
- Assuming that N is directly proportional to P , find an equation of variation.
 - Find N when $P = 200,000$.
 - Find P when $N = 367,200$.
88. **Median age of women at first marriage.** In general, people in our society are marrying at a later age. The median age, $A(t)$, of women at first marriage can be approximated by the linear function
- $$A(t) = 0.08t + 19.7,$$
- where t is the number of years after 1950. Thus, $A(0)$ is the median age of women at first marriage in 1950, $A(50)$ is the median age in 2000, and so on.
- Find $A(0)$, $A(1)$, $A(10)$, $A(30)$, and $A(50)$.
 - What was the median age of women at first marriage in 2008?
 - Graph $A(t)$.

SYNTHESIS

89. Explain and compare the situations in which you would use the slope–intercept equation rather than the point–slope equation.
90. Discuss and relate the concepts of fixed cost, total cost, total revenue, and total profit.
91. **Business: daily sales.** Match each sentence below with the most appropriate of the following graphs (I, II, III, or IV).



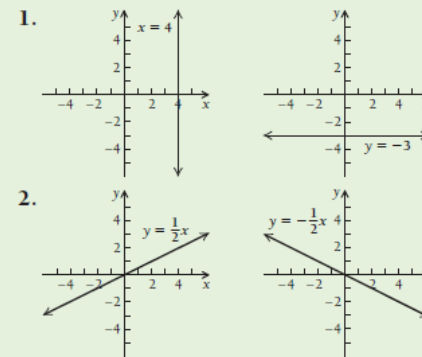
92. **Business: depreciation.** A large crane is being depreciated according to the model $V(t) = 900 - 60t$, where $V(t)$ is measured in thousands of dollars and t is the number of years since 2005. If the crane is to be depreciated until its value is \$0, what is the domain of the depreciation model?

TECHNOLOGY CONNECTION

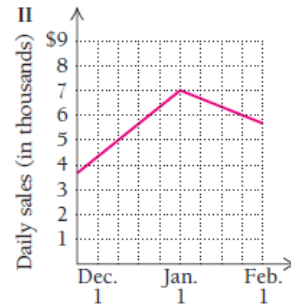
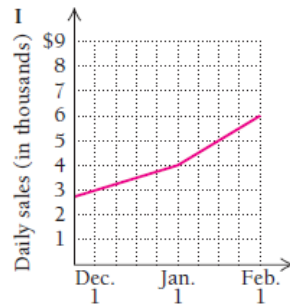


93. Graph some of the total-revenue, total-cost, and total-profit functions in this exercise set using the same set of axes. Identify regions of profit and loss.

Answers to Quick Checks



- a) After January 1, daily sales continued to rise, but at a slower rate.
- b) After January 1, sales decreased faster than they ever grew.
- c) The rate of growth in daily sales doubled after January 1.
- d) After January 1, daily sales decreased at half the rate that they grew in December.



3. $m = \frac{1}{2}$, y-intercept: $\left(0, -\frac{7}{6}\right)$

4. $y = -\frac{2}{3}x + 4$ 5. 7

6. (a) $C(x) = 20x + 20,000$; $R(x) = 80x$;
 $P(x) = R(x) - C(x) = 60x - 20,000$ (b) 333 suits