CHAPTER 5

Computing area under a curve, like the area of the region spanned by the scaffolding under the roller coaster track, is an application of integration.

# INTEGRATION

## SECTION 5.1 Antidifferentiation: The Indefinite Integral

How can a known rate of inflation be used to determine future prices? What is the velocity of an object moving along a straight line with known acceleration? How can knowing the rate at which a population is changing be used to predict future population levels? In all these situations, the derivative (rate of change) of a quantity is known and the quantity itself is required. Here is the terminology we will use in connection with obtaining a function from its derivative.

**Antidifferentiation** ■ A function $F(x)$ is said to be an *antiderivative* of $f(x)$ if

$$F'(x) = f(x)$$

for every $x$ in the domain of $f(x)$. The process of finding antiderivatives is called *antidifferentiation* or *indefinite integration.*

**NOTE** Sometimes we write the equation

$$F'(x) = f(x)$$

as

$$\frac{dF}{dx} = f(x) \quad ■$$

Later in this section, you will learn techniques you can use to find antiderivatives. Once you have found what you believe to be an antiderivative of a function, you can always check your answer by differentiating. You should get the original function back. Here is an example.

### EXAMPLE 5.1.1

Verify that $F(x) = \frac{1}{3}x^3 + 5x + 2$ is an antiderivative of $f(x) = x^2 + 5$.

**Solution**

$F(x)$ is an antiderivative of $f(x)$ if and only if $F'(x) = f(x)$. Differentiate $F$ and you will find that

$$\begin{aligned} F'(x) &= \frac{1}{3}(3x^2) + 5 \\ &= x^2 + 5 = f(x) \end{aligned}$$

as required.

### The General Antiderivative of a Function

A function has more than one antiderivative. For example, one antiderivative of the function $f(x) = 3x^2$ is $F(x) = x^3$, since

$$F'(x) = 3x^2 = f(x)$$

but so are $x^3 + 12$ and $x^3 - 5$ and $x^3 + \pi$, since

$$\frac{d}{dx}(x^3 + 12) = 3x^2 \qquad \frac{d}{dx}(x^3 - 5) = 3x^2 \qquad \frac{d}{dx}(x^3 + \pi) = 3x^2$$

In general, if $F$ is one antiderivative of $f$, then so is any function of the form $G(x) = F(x) + C$, for constant $C$ since

$$\begin{aligned} G'(x) &= [F(x) + C]' \\ &= F'(x) + C' && \text{sum rule for derivatives} \\ &= F'(x) + 0 && \text{derivative of a constant is 0} \\ &= f(x) && \text{since } F \text{ is an antiderivative of } f \end{aligned}$$

Conversely, it can be shown that if $F$ and $G$ are both antiderivatives of $f$, then $G(x) = F(x) + C$, for some constant $C$ (Exercise 64). To summarize:

**Fundamental Property of Antiderivatives** ■ If $F(x)$ is an antiderivative of the continuous function $f(x)$, then any other antiderivative of $f(x)$ has the form $G(x) = F(x) + C$ for some constant $C$.

There is a simple geometric interpretation for the fundamental property of antiderivatives. If $F$ and $G$ are both antiderivatives of $f$, then

$$G'(x) = F'(x) = f(x)$$

This means that the slope $F'(x)$ of the tangent line to $y = F(x)$ at the point $(x, F(x))$ is the same as the slope $G'(x)$ of the tangent line to $y = G(x)$ at $(x, G(x))$. Since the slopes are equal, it follows that the tangent lines at $(x, F(x))$ and $(x, G(x))$ are parallel, as shown in Figure 5.1a. Since this is true for all $x$, the entire curve $y = G(x)$ must be parallel to the curve $y = F(x)$, so that

$$y = G(x) = F(x) + C$$

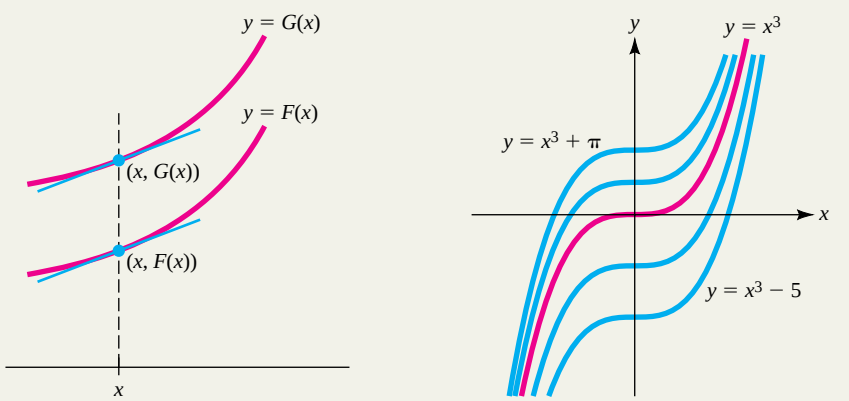
In general, the collection of graphs of all antiderivatives of a given function $f$ is a family of parallel curves that are vertical translations of one another. This is illustrated in Figure 5.1b for the family of antiderivatives of $f(x) = 3x^2$.

***Just-In-Time* REVIEW**

Recall that two lines are parallel if and only if their slopes are equal.

**EXPLORE!**

Store the function $F(x) = x^3$ into Y1 of the equation editor in a bold graphing style. Generate a family of vertical transformations Y2 = Y1 + L1, where L1 is a list of constants, {−4, −2, 2, 4}. Use the graphing window [−4.7, 4.7]1 by [−6, 6]1. What do you observe about the slopes of all these curves at $x = 1$?




(a) If $F'(x) = G'(x)$, the tangent lines at $(x, F(x))$ and $(x, G(x))$ are parallel

(b) Graphs of some members of the family of antiderivatives of $f(x) = 3x^2$

**FIGURE 5.1** Graphs of antiderivatives of a function $f$ form a family of parallel curves.

## The Indefinite Integral

You have just seen that if $F(x)$ is one antiderivative of the continuous function $f(x)$, then all such antiderivatives may be characterized by $F(x) + C$ for constant $C$. The family of all antiderivatives of $f(x)$ is written

$$\int f(x)\,dx = F(x) + C$$

and is called the **indefinite integral** of $f(x)$. The integral is "indefinite" because it involves a constant $C$ that can take on any value. In Section 5.3, we introduce a **definite integral** that has a specific numerical value and is used to represent a variety of quantities, such as area, average value, present value of an income flow, and cardiac output, to name a few. The connection between definite and indefinite integrals is made in Section 5.3 through a result so important that it is referred to as the **fundamental theorem of calculus.**

### *Just-In-Time* REVIEW

Recall that differentials were introduced in Section 2.5.

In the context of the indefinite integral $\int f(x)\,dx = F(x) + C$, the **integral symbol** is $\int$, the function $f(x)$ is called the **integrand,** $C$ is the **constant of integration,** and $dx$ is a differential that specifies $x$ as the **variable of integration.** These features are displayed in this diagram for the indefinite integral of $f(x) = 3x^2$:

integrand — constant of integration

$$\int 3x^2\,dx = x^3 + C$$

integral symbol — variable of integration

For any differentiable function $F$, we have

$$\int F'(x)\,dx = F(x) + C$$

since by definition, $F(x)$ is an antiderivative of $F'(x)$. Equivalently,

$$\int \frac{dF}{dx}\,dx = F(x) + C$$

This property of indefinite integrals is especially useful in applied problems where a rate of change $F'(x)$ is given and we wish to find $F(x)$. Several such problems are examined later in this section, in Examples 5.1.4 through 5.1.8.

### EXPLORE!

Most graphing calculators allow the construction of an antiderivative through its numerical integral, **fnInt**(expression, variable, lower limit, upper limit), found in the **MATH** menu. In the equation editor of your calculator write

Y1 = **fnInt(2X, X, {0, 1, 2}, X)**

and graph using an expanded decimal window, [−4.7, 4.7]1 by [−5, 5]1. What do you observe and what is the general form for this family of antiderivatives?

It is useful to remember that if you have performed an indefinite integration calculation that leads you to believe that $\int f(x)\,dx = G(x) + C$, then you can check your calculation by differentiating $G(x)$:

*If $G'(x) = f(x)$, then the integration $\int f(x)\,dx = G(x) + C$ is correct, but if $G'(x)$ is anything other than $f(x)$, you've made a mistake.*

This relationship between differentiation and antidifferentiation enables us to establish these integration rules by "reversing" analogous differentiation rules.

### Rules for Integrating Common Functions

The **constant rule:** $\int k\,dx = kx + C$ for constant $k$

The **power rule:** $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$ for all $n \neq -1$

The **logarithmic rule:** $\int \dfrac{1}{x}\,dx = \ln|x| + C$ for all $x \neq 0$

The **exponential rule:** $\int e^{kx}\,dx = \dfrac{1}{k}e^{kx} + C$ for constant $k \neq 0$

To verify the power rule, it is enough to show that the derivative of $\dfrac{x^{n+1}}{n+1}$ is $x^n$:

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{1}{n+1}[(n+1)x^n] = x^n$$

For the logarithmic rule, if $x > 0$, then $|x| = x$ and

$$\frac{d}{dx}(\ln|x|) = \frac{d}{dx}(\ln x) = \frac{1}{x}$$

If $x < 0$, then $-x > 0$ and $\ln|x| = \ln(-x)$, and it follows from the chain rule that

$$\frac{d}{dx}(\ln|x|) = \frac{d}{dx}[\ln(-x)] = \frac{1}{(-x)}(-1) = \frac{1}{x}$$

Thus, for all $x \neq 0$,

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

so

$$\int \frac{1}{x}dx = \ln|x| + C$$

You are asked to verify the constant rule and exponential rule in Exercise 66.

**NOTE** Notice that the logarithm rule "fills the gap" in the power rule; namely, the case where $n = -1$. You may wish to blend the two rules into this combined form:

$$\int x^n\,dx = \begin{cases} \dfrac{x^{n+1}}{n+1} + C & \text{if } n \neq -1 \\ \ln|x| + C & \text{if } n = -1 \end{cases}$$

**EXPLORE!**

Graph $y = F(x)$, where

$F(x) = \ln|x| = \ln(\text{abs}(x))$

in bold and $f(x) = \dfrac{1}{x}$ in the regular graphing style using a decimal graphing window. At any point $x \neq 0$, show that the derivative of $F(x)$ is equal to the value of $f(x)$ at that particular point, confirming that $F(x)$ is the antiderivative of $f(x)$.

## EXAMPLE 5.1.2

Find these integrals:

**a.** $\int 3\,dx$ **b.** $\int x^{17}\,dx$ **c.** $\int \frac{1}{\sqrt{x}}\,dx$ **d.** $\int e^{-3x}\,dx$

### Solution

**a.** Use the constant rule with $k = 3$: $\int 3\,dx = 3x + C$

**b.** Use the power rule with $n = 17$: $\int x^{17}\,dx = \frac{1}{18}x^{18} + C$

**c.** Use the power rule with $n = -\frac{1}{2}$: Since $n + 1 = \frac{1}{2}$,

$$\int \frac{dx}{\sqrt{x}} = \int x^{-1/2}\,dx = \frac{1}{1/2}x^{1/2} + C = 2\sqrt{x} + C$$

**d.** Use the exponential rule with $k = -3$:

$$\int e^{-3x}\,dx = \frac{1}{-3}e^{-3x} + C$$

Example 5.1.2 illustrates how certain basic functions can be integrated, but what about combinations of functions, such as the polynomial $x^5 + 2x^3 + 7$ or an expression like $5e^{-x} + \sqrt{x}$? Here are algebraic rules that will enable you to handle such expressions in a natural fashion.

### Algebraic Rules for Indefinite Integration

The **constant multiple rule:** $\int kf(x)\,dx = k\int f(x)\,dx$ for constant $k$

The **sum rule:** $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

The **difference rule:** $\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$

To prove the constant multiple rule, note that if $\frac{dF}{dx} = f(x)$, then

$$\frac{d}{dx}[kF(x)] = k\frac{dF}{dx} = kf(x)$$

which means that

$$\int kf(x)\,dx = k\int f(x)\,dx$$

The sum and difference rules can be established in a similar fashion.

## EXAMPLE 5.1.3

Find the following integrals:

**a.** $\displaystyle\int (2x^5 + 8x^3 - 3x^2 + 5)\,dx$

**b.** $\displaystyle\int \left(\frac{x^3 + 2x - 7}{x}\right) dx$

**c.** $\displaystyle\int (3e^{-5t} + \sqrt{t})\,dt$

### Solution

**a.** By using the power rule in conjunction with the sum and difference rules and the multiple rule, you get

$$\begin{aligned}\int (2x^5 + 8x^3 - 3x^2 + 5)\,dx &= 2\int x^5\,dx + 8\int x^3\,dx - 3\int x^2\,dx + \int 5\,dx \\ &= 2\left(\frac{x^6}{6}\right) + 8\left(\frac{x^4}{4}\right) - 3\left(\frac{x^3}{3}\right) + 5x + C \\ &= \frac{1}{3}x^6 + 2x^4 - x^3 + 5x + C\end{aligned}$$

**b.** There is no "quotient rule" for integration, but at least in this case, you can still divide the denominator into the numerator and then integrate using the method in part (a):

$$\begin{aligned}\int \left(\frac{x^3 + 2x - 7}{x}\right) dx &= \int \left(x^2 + 2 - \frac{7}{x}\right) dx \\ &= \frac{1}{3}x^3 + 2x - 7\ln|x| + C\end{aligned}$$

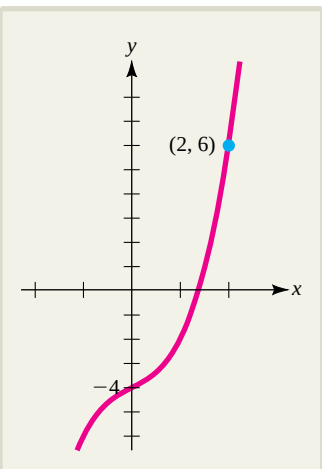
**c.** $$\begin{aligned}\int (3e^{-5t} + \sqrt{t})\,dt &= \int (3e^{-5t} + t^{1/2})\,dt \\ &= 3\left(\frac{1}{-5}e^{-5t}\right) + \frac{1}{3/2}t^{3/2} + C = -\frac{3}{5}e^{-5t} + \frac{2}{3}t^{3/2} + C\end{aligned}$$

## EXPLORE!

Refer to Example 5.1.4. Store the function $f(x) = 3x^2 + 1$ into Y1. Graph using a bold graphing style and the window [0, 2.35]0.5 by [−2, 12]1. Place into Y2 the family of antiderivatives

$$F(x) = x^3 + x + \mathbf{L1}$$

where L1 is the list of integer values −5 to 5. Which of these antiderivatives passes through the point (2, 6)? Repeat this exercise for $f(x) = 3x^2 - 2$.




The graph of $y = x^3 + x - 4$.

## EXAMPLE 5.1.4

Find the function $f(x)$ whose tangent has slope $3x^2 + 1$ for each value of $x$ and whose graph passes through the point (2, 6).

### Solution

The slope of the tangent at each point $(x, f(x))$ is the derivative $f'(x)$. Thus,

$$f'(x) = 3x^2 + 1$$

and so $f(x)$ is the antiderivative

$$f(x) = \int f'(x)\,dx = \int (3x^2 + 1)\,dx = x^3 + x + C$$

To find $C$, use the fact that the graph of $f$ passes through (2, 6). That is, substitute $x = 2$ and $f(2) = 6$ into the equation for $f(x)$ and solve for $C$ to get

$$6 = (2)^3 + 2 + C \qquad \text{or} \qquad C = -4$$

Thus, the desired function is $f(x) = x^3 + x - 4$. The graph of this function is shown in the accompanying figure.

## Applied Initial Value Problems

A **differential equation** is an equation that involves differentials or derivatives. Such equations are of great importance in modeling and occur in a variety of applications. An **initial value problem** is a problem that involves solving a differential equation subject to a specified initial condition. For instance, in Example 5.1.4, we were required to find $y = f(x)$ so that

$$\frac{dy}{dx} = 3x^2 + 1 \quad \text{subject to the condition } y = 6 \text{ when } x = 2$$

We solved this initial value problem by finding the antiderivative

$$y = \int (3x^2 + 1)\, dx = x^3 + x + C$$

and then using the initial condition to evaluate $C$. The same approach is used in Examples 5.1.5 through 5.1.8 to solve a selection of applied initial value problems from business, economics, biology, and physics. Similar initial value problems appear in examples and exercises throughout this chapter.

### EXAMPLE 5.1.5

A manufacturer has found that marginal cost is $3q^2 - 60q + 400$ dollars per unit when $q$ units have been produced. The total cost of producing the first 2 units is \$900. What is the total cost of producing the first 5 units?

#### Solution

Recall that the marginal cost is the derivative of the total cost function $C(q)$. Thus,

$$\frac{dC}{dq} = 3q^2 - 60q + 400$$

and so $C(q)$ must be the antiderivative

$$C(q) = \int \frac{dC}{dq}\, dq = \int (3q^2 - 60q + 400)\, dq = q^3 - 30q^2 + 400q + K$$

for some constant $K$. (The letter $K$ was used for the constant to avoid confusion with the cost function $C$.)

The value of $K$ is determined by the fact that $C(2) = 900$. In particular,

$$900 = (2)^3 - 30(2)^2 + 400(2) + K \qquad \text{or} \qquad K = 212$$

Hence,

$$C(q) = q^3 - 30q^2 + 400q + 212$$

and the cost of producing the first 5 units is

$$C(5) = (5)^3 - 30(5)^2 + 400(5) + 212 = \$1{,}587$$

**EXPLORE!**

Graph the function $P(t)$ from Example 5.1.6, using the window [0, 23.5]5 by [175,000, 225,000]25,000. Display the population 9 hours from now. When will the population hit 300,000?

## EXAMPLE 5.1.6

The population $P(t)$ of a bacterial colony $t$ hours after observation begins is found to be changing at the rate

$$\frac{dP}{dt} = 200e^{0.1t} + 150e^{-0.03t}$$

If the population was 200,000 bacteria when observations began, what will the population be 12 hours later?

### Solution

The population $P(t)$ is found by antidifferentiating $\frac{dP}{dt}$ as follows:

$$\begin{aligned} P(t) &= \int \frac{dP}{dt}\,dt = \int (200e^{0.1t} + 150e^{-0.03t})\,dt \\ &= \frac{200e^{0.1t}}{0.1} + \frac{150e^{-0.03t}}{-0.03} + C \qquad \text{exponential and sum rules} \\ &= 2{,}000\,e^{0.1t} - 5{,}000\,e^{-0.03t} + C \end{aligned}$$

Since the population is 200,000 when $t = 0$, we have

$$\begin{aligned} P(0) = 200{,}000 &= 2{,}000\,e^0 - 5{,}000\,e^0 + C \\ &= -3{,}000 + C \end{aligned}$$

so $C = 203{,}000$ and

$$P(t) = 2{,}000e^{0.1t} - 5{,}000e^{-0.03t} + 203{,}000$$

Thus, after 12 hours, the population is

$$\begin{aligned} P(12) &= 2{,}000e^{0.1(12)} - 5{,}000e^{-0.03(12)} + 203{,}000 \\ &\approx 206{,}152 \end{aligned}$$

## EXAMPLE 5.1.7

A retailer receives a shipment of 10,000 kilograms of rice that will be used up over a 5-month period at the constant rate of 2,000 kilograms per month. If storage costs are 1 cent per kilogram per month, how much will the retailer pay in storage costs over the next 5 months?

### Solution

Let $S(t)$ denote the total storage cost (in dollars) over $t$ months. Since the rice is used up at a constant rate of 2,000 kilograms per month, the number of kilograms of rice in storage after $t$ months is $10{,}000 - 2{,}000t$. Therefore, since storage costs are 1 cent per kilogram per month, the rate of change of the storage cost with respect to time is

$$\frac{dS}{dt} = \begin{pmatrix}\text{monthly cost}\\ \text{per kilogram}\end{pmatrix}\begin{pmatrix}\text{number of}\\ \text{kilograms}\end{pmatrix} = 0.01(10{,}000 - 2{,}000t)$$

It follows that $S(t)$ is an antiderivative of

$$0.01(10{,}000 - 2{,}000t) = 100 - 20t$$

That is,
$$S(t) = \int \frac{dS}{dt}\,dt = \int (100 - 20t)\,dt$$
$$= 100t - 10t^2 + C$$

for some constant $C$. To determine $C$, use the fact that at the time the shipment arrives (when $t = 0$) there is no cost, so that

$$0 = 100(0) - 10(0)^2 + C \qquad \text{or} \qquad C = 0$$

Hence,
$$S(t) = 100t - 10t^2$$

and the total storage cost over the next 5 months will be

$$S(5) = 100(5) - 10(5)^2 = \$250$$

## Motion along a Line

Recall from Section 2.2 that if an object moving along a straight line is at the position $s(t)$ at time $t$, then its velocity is given by $v = \frac{ds}{dt}$ and its acceleration by $a = \frac{dv}{dt}$. Turning things around, if the acceleration of the object is given, then its velocity and position can be found by integration. Here is an example.

**EXPLORE!**

Refer to Example 5.1.8. Graph the position function $s(t)$ in the equation editor of your calculator as Y1 $= -11x^2 + 66x$, using the window [0, 9.4]1 by [0, 200]10. Locate the stopping time and the corresponding position on the graph. Work the problem again for a car traveling 60 mph (88 ft/sec). In this case, what is happening at the 3-sec mark?

### EXAMPLE 5.1.8

A car is traveling along a straight, level road at 45 miles per hour (66 feet per second) when the driver is forced to apply the brakes to avoid an accident. If the brakes supply a constant deceleration of 22 ft/sec$^2$ (feet per second, per second), how far does the car travel before coming to a complete stop?

#### Solution

Let $s(t)$ denote the distance traveled by the car in $t$ seconds after the brakes are applied. Since the car decelerates at 22 ft/sec$^2$, it follows that $a(t) = -22$; that is,

$$\frac{dv}{dt} = a(t) = -22$$

Integrating, you find that the velocity at time $t$ is given by

$$v(t) = \int \frac{dv}{dt}\,dt = \int -22\,dt = -22t + C_1$$

To evaluate $C_1$, note that $v = 66$ when $t = 0$ so that

$$66 = v(0) = -22(0) + C_1$$

and $C_1 = 66$. Thus, the velocity at time $t$ is $v(t) = -22t + 66$.

Next, to find the distance $s(t)$, begin with the fact that

$$\frac{ds}{dt} = v(t) = -22t + 66$$

and use integration to show that

$$s(t) = \int \frac{ds}{dt}\,dt = \int (-22t + 66)\,dt = -11t^2 + 66t + C_2$$

Since $s(0) = 0$ (do you see why?), it follows that $C_2 = 0$ and

$$s(t) = -11t^2 + 66t$$

Finally, to find the stopping distance, note that the car stops when $v(t) = 0$, and this occurs when

$$v(t) = -22t + 66 = 0$$

Solving this equation, you find that the car stops after 3 seconds of deceleration, and in that time it has traveled

$$s(3) = -11(3)^2 + 66(3) = 99 \text{ feet}$$

## EXERCISES ■ 5.1

*In Exercises 1 through 30, find the indicated integral. Check your answers by differentiation.*

**1.** $\int -3\,dx$

**2.** $\int dx$

**3.** $\int x^5\,dx$

**4.** $\int \sqrt{t}\,dt$

**5.** $\int \frac{1}{x^2}\,dx$

**6.** $\int 3e^x\,dx$

**7.** $\int \frac{2}{\sqrt{t}}\,dt$

**8.** $\int x^{-0.3}\,dx$

**9.** $\int u^{-2/5}\,du$

**10.** $\int \left(\frac{1}{x^2} - \frac{1}{x^3}\right) dx$

**11.** $\int (3t^2 - \sqrt{5t} + 2)\,dt$

**12.** $\int (x^{1/3} - 3x^{-2/3} + 6)\,dx$

**13.** $\int (3\sqrt{y} - 2y^{-3})\,dy$

**14.** $\int \left(\frac{1}{2y} - \frac{2}{y^2} + \frac{3}{\sqrt{y}}\right) dy$

**15.** $\int \left(\frac{e^x}{2} + x\sqrt{x}\right) dx$

**16.** $\int \left(\sqrt{x^3} - \frac{1}{2\sqrt{x}} + \sqrt{2}\right) dx$

**17.** $\int u^{1.1}\left(\frac{1}{3u} - 1\right) du$

**18.** $\int \left(2e^u + \frac{6}{u} + \ln 2\right) du$

**19.** $\int \left(\frac{x^2 + 2x + 1}{x^2}\right) dx$

**20.** $\int \frac{x^2 + 3x - 2}{\sqrt{x}}\,dx$

**21.** $\int (x^3 - 2x^2)\left(\frac{1}{x} - 5\right) dx$

**22.** $\int y^3\left(2y + \frac{1}{y}\right) dy$

**23.** $\int \sqrt{t}(t^2 - 1)\,dt$

**24.** $\int x(2x + 1)^2\,dx$

**25.** $\int (e^t + 1)^2\,dt$

**26.** $\int e^{-0.02t}(e^{-0.13t} + 4)\,dt$

**27.** $\int \left(\frac{1}{3y} - \frac{5}{\sqrt{y}} + e^{-y/2}\right) dy$

**28.** $\int \frac{1}{x}(x + 1)^2\,dx$

**29.** $\int t^{-1/2}(t^2 - t + 2)\,dt$

**30.** $\int \ln(e^{-x^2})\,dx$

*In Exercises 31 through 34, solve the given initial value problem for $y = f(x)$.*

**31.** $\frac{dy}{dx} = 3x - 2$ where $y = 2$ when $x = -1$

**32.** $\frac{dy}{dx} = e^{-x}$ where $y = 3$ when $x = 0$

**33.** $\frac{dy}{dx} = \frac{2}{x} - \frac{1}{x^2}$ where $y = -1$ when $x = 1$

**34.** $\frac{dy}{dx} = \frac{x + 1}{\sqrt{x}}$ where $y = 5$ when $x = 4$

*In Exercises 35 through 42, the slope $f'(x)$ at each point $(x, y)$ on a curve $y = f(x)$ is given along with a particular point $(a, b)$ on the curve. Use this information to find $f(x)$.*

**35.** $f'(x) = 4x + 1$; $(1, 2)$

**36.** $f'(x) = 3 - 2x$; $(0, -1)$

**37.** $f'(x) = -x(x + 1)$; $(-1, 5)$

**38.** $f'(x) = 3x^2 + 6x - 2$; $(0, 6)$

**39.** $f'(x) = x^3 - \frac{2}{x^2} + 2$; $(1, 3)$

**40.** $f'(x) = x^{-1/2} + x$; $(1, 2)$

**41.** $f'(x) = e^{-x} + x^2$; $(0, 4)$

**42.** $f'(x) = \frac{3}{x} - 4$; $(1, 0)$

**43.** **MARGINAL COST** A manufacturer estimates that the marginal cost of producing $q$ units of a certain commodity is $C'(q) = 3q^2 - 24q + 48$ dollars per unit. If the cost of producing 10 units is \$5,000, what is the cost of producing 30 units?

**44.** **MARGINAL REVENUE** The marginal revenue derived from producing $q$ units of a certain commodity is $R'(q) = 4q - 1.2q^2$ dollars per unit. If the revenue derived from producing 20 units is \$30,000, how much revenue should be expected from producing 40 units?

**45.** **MARGINAL PROFIT** A manufacturer estimates marginal revenue to be $R'(q) = 100q^{-1/2}$ dollars per unit when the level of production is $q$ units. The corresponding marginal cost has been found to be $0.4q$ dollars per unit. Suppose the manufacturer's profit is \$520 when the level of production is 16 units. What is the manufacturer's profit when the level of production is 25 units?

**46.** **SALES** The monthly sales at an import store are currently \$10,000 but are expected to be declining at the rate of

$$S'(t) = -10t^{2/5} \quad \text{dollars per month}$$

$t$ months from now. The store is profitable as long as the sales level is above \$8,000 per month.

**a.** Find a formula for the expected sales in $t$ months.

**b.** What sales figure should be expected 2 years from now?

**c.** For how many months will the store remain profitable?

**47.** **ADVERTISING** After initiating an advertising campaign in an urban area, a satellite dish provider estimates that the number of new subscribers will grow at a rate given by

$$N'(t) = 154t^{2/3} + 37 \quad \text{subscribers per month}$$

where $t$ is the number of months after the advertising begins. How many new subscribers should be expected 8 months from now?

**48.** **TREE GROWTH** An environmentalist finds that a certain type of tree grows in such a way that its height $h(t)$ after $t$ years is changing at the rate of

$$h'(t) = 0.2t^{2/3} + \sqrt{t} \quad \text{ft/yr}$$

If the tree was 2 feet tall when it was planted, how tall will it be in 27 years?

**49.** **POPULATION GROWTH** It is estimated that $t$ months from now the population of a certain town will be increasing at the rate of $4 + 5t^{2/3}$ people per month. If the current population is 10,000, what will be the population 8 months from now?

**50.** **NET CHANGE IN A BIOMASS** A biomass is growing at the rate of $M'(t) = 0.5e^{0.2t}$ g/hr. By how much does the mass change during the second hour?

**51.** **LEARNING** Bob is taking a learning test in which the time he takes to memorize items from a given list is recorded. Let $M(t)$ be the number of items he can memorize in $t$ minutes. His learning rate is found to be

$$M'(t) = 0.4t - 0.005t^2$$

**a.** How many items can Bob memorize during the first 10 minutes?

**b.** How many additional items can he memorize during the next 10 minutes (from time $t = 10$ to $t = 20$)?

**52.** **ENDANGERED SPECIES** A conservationist finds that the population $P(t)$ of a certain endangered species is growing at a rate given by $P'(t) = 0.51e^{-0.03t}$, where $t$ is the number of years after records began to be kept.

**a.** If the population is $P_0 = 500$ now (at time $t = 0$), what will it be in 10 years?

**b.** Read an article on endangered species and write a paragraph on the use of mathematical models in studying populations of such species.*

**53.** **DEFROSTING** A roast is removed from the freezer of a refrigerator and left on the counter to defrost. The temperature of the roast was $-4°C$ when it was removed from the freezer and $t$ hours later, was increasing at the rate of

$$T'(t) = 7e^{-0.35t} \quad °C/hr$$

**a.** Find a formula for the temperature of the roast after $t$ hours.

**b.** What is the temperature after 2 hours?

**c.** Assume the roast is defrosted when its temperature reaches 10°C. How long does it take for the roast to defrost?

**54.** **MARGINAL REVENUE** Suppose it has been determined that the marginal revenue associated with the production of $x$ units of a particular commodity is $R'(x) = 240 - 4x$ dollars per unit. What is the revenue function $R(x)$? You may assume $R(0) = 0$. What price will be paid for each unit when the level of production is $x = 5$ units?

**55.** **MARGINAL PROFIT** The marginal profit of a certain commodity is $P'(q) = 100 - 2q$ when $q$ units are produced. When 10 units are produced, the profit is \$700.

**a.** Find the profit function $P(q)$.

**b.** What production level $q$ results in maximum profit? What is the maximum profit?

**56.** **PRODUCTION** At a certain factory, when $K$ thousand dollars is invested in the plant, the production $Q$ is changing at a rate given by

$$Q'(K) = 200K^{-2/3}$$

units per thousand dollars invested. When \$8,000 is invested, the level of production is 5,500 units.

**a.** Find a formula for the level of production $Q$ to be expected when $K$ thousand dollars is invested.

**b.** How many units will be produced when \$27,000 is invested?

**c.** What capital investment $K$ is required to produce 7,000 units?

**57.** **MARGINAL PROPENSITY TO CONSUME** Suppose the consumption function for a particular country is $c(x)$, where $x$ is national disposable income. Then the **marginal propensity to consume** is $c'(x)$. Suppose $x$ and $c$ are both measured in billions of dollars and

$$c'(x) = 0.9 + 0.3\sqrt{x}$$

If consumption is 10 billion dollars when $x = 0$, find $c(x)$.

**58.** **MARGINAL ANALYSIS** A manufacturer estimates marginal revenue to be $200q^{-1/2}$ dollars per unit when the level of production is $q$ units. The corresponding marginal cost has been found to be $0.4q$ dollars per unit. If the manufacturer's profit is \$2,000 when the level of production is 25 units, what is the profit when the level of production is 36 units?

**59.** **SPY STORY** Our spy, intent on avenging the death of Siggy Leiter (Exercise 67 in Section 4.2), is driving a sports car toward the lair of the fiend who killed his friend. To remain as inconspicuous as possible, he is traveling at the legal speed of 60 mph (88 feet per second) when suddenly, he sees a camel in the road, 199 feet in front of him. It takes him 00.7 seconds to react to the crisis. Then he hits the brakes, and the car decelerates at the constant rate of 28 ft/sec$^2$ (28 feet per second, per second). Does he stop before hitting the camel?

**60.** **CANCER THERAPY** A new medical procedure is applied to a cancerous tumor with volume 30 cm$^3$, and $t$ days later the volume is found to be changing at the rate

$$V'(t) = 0.15 - 0.09e^{0.006t} \quad cm^3/day$$

*You may wish to begin your research with the journal *Ecology*.

**a.** Find a formula for the volume of the tumor after $t$ days.
**b.** What is the volume after 60 days? After 120 days?
**c.** For the procedure to be successful, it should take no longer than 90 days for the tumor to begin to shrink. Based on this criterion, does the procedure succeed?

**61.** **LEARNING** Let $f(x)$ represent the total number of items a subject has memorized $x$ minutes after being presented with a long list of items to learn. Psychologists refer to the graph of $y = f(x)$ as a **learning curve** and to $f'(x)$ as the **learning rate.** The time of **peak efficiency** is the time when the learning rate is maximized. Suppose the learning rate is

$$f'(x) = 0.1(10 + 12x - 0.6x^2) \quad \text{for } 0 \le x \le 25$$

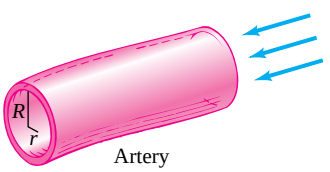
**a.** When does peak efficiency occur? What is the learning rate at peak efficiency?
**b.** What is $f(x)$? You may assume that $f(0) = 0$.
**c.** What is the largest number of items memorized by the subject?

**62.** **CORRECTION FACILITY MANAGEMENT** Statistics compiled by the local department of corrections indicate that $x$ years from now the number of inmates in county prisons will be increasing at the rate of $280e^{0.2x}$ per year. Currently, 2,000 inmates are housed in county prisons. How many inmates should the county expect 10 years from now?

**63.** **FLOW OF BLOOD** One of Poiseuille's laws for the flow of blood in an artery says that if $v(r)$ is the velocity of flow $r$ cm from the central axis of the artery, then the velocity decreases at a rate proportional to $r$. That is,

$$v'(r) = -ar$$

where $a$ is a positive constant.* Find an expression for $v(r)$. Assume $v(R) = 0$, where $R$ is the radius of the artery.




**EXERCISE 63**

*E. Batschelet, *Introduction to Mathematics for Life Scientists*, 2nd ed., New York: Springer-Verlag, 1979, pp. 101–103.

**64.** If $H'(x) = 0$ for all real numbers $x$, what must be true about the graph of $H(x)$? Explain how your observation can be used to show that if $G'(x) = F'(x)$ for all $x$, then $G(x) = F(x) + C$ for constant $C$.

**65.** **DISTANCE AND VELOCITY** An object is moving so that its velocity after $t$ minutes is $v(t) = 3 + 2t + 6t^2$ meters per minute. How far does the object travel during the second minute?

**66.** **a.** Prove the constant rule: $\int k\,dx = kx + C$.

**b.** Prove the exponential rule: $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$.

**67.** What is $\int b^x\,dx$ for base $b$ ($b > 0$, $b \neq 1$)? [*Hint:* Recall that $b^x = e^{x \ln b}$.]

**68.** It is estimated that $x$ months from now, the population of a certain town will be changing at the rate of $P'(x) = 2 + 1.5\sqrt{x}$ people per month. The current population is 5,000.
**a.** Find a function $P(x)$ that satisfies these conditions. Use the graphing utility of your calculator to graph this function.
**b.** Use **TRACE** and **ZOOM** to determine the level of population 9 months from now. When will the population be 7,590?
**c.** Suppose the current population were 2,000 (instead of 5,000). Sketch the graph of $P(x)$ with this assumption. Then sketch the graph of $P(x)$ assuming current populations of 4,000 and 6,000. What is the difference between the graphs?

**69.** A car traveling at 67 ft/sec decelerates at the constant rate of 23 ft/sec$^2$ when the brakes are applied.
**a.** Find the velocity $v(t)$ of the car $t$ seconds after the brakes are applied. Then find its distance $s(t)$ from the point where the brakes are applied.
**b.** Use the graphing utility of your calculator to sketch the graphs of $v(t)$ and $s(t)$ on the same screen (use [0, 5]1 by [0, 200]10).
**c.** Use **TRACE** and **ZOOM** to determine when the car comes to a complete stop and how far it travels in that time. How fast is the car traveling when it has traveled 45 feet?

## SECTION 5.2 Integration by Substitution

The majority of functions that occur in practical situations can be differentiated by applying rules and formulas such as those you learned in Chapter 2. Integration, however, is at least as much an art as a science, and many integrals that appear deceptively simple may actually require a special technique or clever insight.

For example, we easily find that

$$\int x^7\,dx = \frac{1}{8}x^8 + C$$

by applying the power rule, but suppose we wish to compute

$$\int (3x+5)^7\,dx$$

We could proceed by expanding the integrand $(3x+5)^7$ and then integrating term by term, but the algebra involved in this approach is daunting. Instead, we make the change of variable

$$u = 3x + 5 \qquad \text{so that} \qquad du = 3\,dx \quad \text{or} \quad dx = \frac{1}{3}\,du$$

**Just-In-Time REVIEW**

Recall that the differential of $y = f(x)$ is $dy = f'(x)\,dx$.

Then, by substituting these quantities into the given integral, we get

$$\begin{aligned}\int (3x+5)^7\,dx &= \int u^7\left(\frac{1}{3}\,du\right)\\ &= \frac{1}{3}\left(\frac{1}{8}u^8\right) + C = \frac{1}{24}u^8 + C && \text{power rule}\\ &= \frac{1}{24}(3x+5)^8 + C && \text{since } u = 3x+5\end{aligned}$$

We can check this computation by differentiating using the chain rule (Section 2.4):

$$\frac{d}{dx}\left[\frac{1}{24}(3x+5)^8\right] = \frac{1}{24}[8(3x+5)^7(3)] = (3x+5)^7$$

which verifies that $\frac{1}{24}(3x+5)^8$ is indeed an antiderivative of $(3x+5)^7$.

The change of variable procedure we have just demonstrated is called **integration by substitution,** and it amounts to reversing the chain rule for differentiation. To see why, consider an integral that can be written as

$$\int f(x)\,dx = \int g(u(x))\,u'(x)\,dx$$

Suppose $G$ is an antiderivative of $g$, so that $G' = g$. Then, according to the chain rule

$$\begin{aligned}\frac{d}{dx}[G(u(x))] &= G'(u(x))\,u'(x)\\ &= g(u(x))\,u'(x) && \text{since } G' = g\end{aligned}$$

Therefore, by integrating both sides of this equation with respect to $x$, we find that

$$\begin{aligned}\int f(x)\,dx &= \int g(u(x))\,u'(x)\,dx \\ &= \int \left(\frac{d}{dx}[G(u(x))]\right) dx \\ &= G(u(x)) + C \qquad \text{since } \textstyle\int G' = G\end{aligned}$$

In other words, once we have an antiderivative for $g(u)$, we also have one for $f(x)$.

A useful device for remembering the substitution procedure is to think of $u = u(x)$ as a change of variable whose differential $du = u'(x)\,dx$ can be manipulated algebraically. Then

$$\begin{aligned}\int f(x)\,dx &= \int g(u(x))\,u'(x)\,dx \\ &= \int g(u)\,du \qquad \text{substitute du for } u'(x)\,dx \\ &= G(u) + C \qquad \text{where } G \text{ is an antiderivative of } g \\ &= G(u(x)) + C \qquad \text{substitute } u(x) \text{ for } u\end{aligned}$$

Here is a step-by-step procedure for integrating by substitution.

### Using Substitution to Integrate $\int f(x)\,dx$

**Step 1.** Choose a substitution $u = u(x)$ that "simplifies" the integrand $f(x)$.

**Step 2.** Express the entire integral in terms of $u$ and $du = u'(x)\,dx$. This means that *all* terms involving $x$ and $dx$ must be transformed to terms involving $u$ and $du$.

**Step 3.** When step 2 is complete, the given integral should have the form

$$\int f(x)\,dx = \int g(u)\,du$$

If possible, evaluate this transformed integral by finding an antiderivative $G(u)$ for $g(u)$.

**Step 4.** Replace $u$ by $u(x)$ in $G(u)$ to obtain an antiderivative $G(u(x))$ for $f(x)$, so that

$$\int f(x)\,dx = G(u(x)) + C$$

An old saying goes, "The first step in making rabbit stew is to catch a rabbit." Likewise, the first step in integrating by substitution is to find a suitable change of variable $u = u(x)$ that simplifies the integrand of the given integral $\int f(x)\,dx$ without adding undesired complexity when $dx$ is replaced by $du = u'(x)\,dx$. Here are a few guidelines for choosing $u(x)$:

1. If possible, try to choose $u$ so that $u'(x)$ is part of the integrand $f(x)$.
2. Consider choosing $u$ as the part of the integrand that makes $f(x)$ difficult to integrate directly, such as the quantity inside a radical, the denominator of a fraction, or the exponent of an exponential function.
3. Don't "oversubstitute." For instance, in our introductory example $\int(3x + 5)^7\,dx$, a common mistake is to use $u = (3x + 5)^7$. This certainly simplifies the integrand, but then $du = 7(3x + 5)^6(3)\,dx$, and you are left with a transformed integral that is more complicated than the original.
4. Persevere. If you try a substitution that does not result in a transformed integral you can evaluate, try a different substitution.

Examples 5.2.1 through 5.2.6 illustrate how substitutions are chosen and used in various kinds of integrals.

## EXAMPLE 5.2.1

Find $\displaystyle\int \sqrt{2x + 7}\,dx$.

### Solution

We choose $u = 2x + 7$ and obtain

$$du = 2\,dx \qquad \text{so that} \qquad dx = \frac{1}{2}\,du$$

Then the integral becomes

$$\begin{aligned}
\int \sqrt{2x + 7}\,dx &= \int \sqrt{u}\left(\frac{1}{2}\,du\right) \\
&= \frac{1}{2}\int u^{1/2}\,du && \text{since } \sqrt{u} = u^{1/2} \\
&= \frac{1}{2}\,\frac{u^{3/2}}{3/2} + C = \frac{1}{3}u^{3/2} + C && \text{power rule} \\
&= \frac{1}{3}(2x + 7)^{3/2} + C && \text{substitute } 2x + 7 \text{ for } u
\end{aligned}$$

## EXAMPLE 5.2.2

Find $\displaystyle\int 8x(4x^2 - 3)^5\,dx$.

### Solution

First, note that the integrand $8x(4x^2 - 3)^5$ is a product in which one of the factors, $8x$, is the derivative of an expression, $4x^2 - 3$, that appears in the other factor. This suggests that you make the substitution

$$u = 4x^2 - 3 \qquad \text{with} \qquad du = 4(2x\,dx) = 8x\,dx$$

to obtain

$$\begin{aligned}\int 8x(4x^2-3)^5\,dx &= \int (4x^2-3)^5(8x\,dx)\\ &= \int u^5\,dx\\ &= \frac{1}{6}u^6 + C && \text{power rule}\\ &= \frac{1}{6}(4x^2-3)^6 + C && \text{substitute } 4x^2-3 \text{ for } u\end{aligned}$$

## EXAMPLE 5.2.3

Find $\int x^3 e^{x^4+2}\,dx$.

### Solution

If the integrand of an integral contains an exponential function, it is often useful to substitute for the exponent. In this case, we choose

$$u = x^4 + 2 \qquad \text{so that} \qquad du = 4x^3\,dx$$

and

$$\begin{aligned}\int x^3 e^{x^4+2}\,dx &= \int e^{x^4+2}(x^3\,dx)\\ &= \int e^u\left(\frac{1}{4}\,du\right) && \text{since } du = 4x^3\,dx\\ &= \frac{1}{4}e^u + C && \text{exponential rule}\\ &= \frac{1}{4}e^{x^4+2} + C && \text{substitute } x^4+2 \text{ for } u\end{aligned}$$

## EXAMPLE 5.2.4

Find $\int \frac{x}{x-1}\,dx$.

### Solution

Following our guidelines, we substitute for the denominator of the integrand, so that $u = x - 1$ and $du = dx$. Since $u = x - 1$, we also have $x = u + 1$. Thus,

$$\begin{aligned}\int \frac{x}{x-1}\,dx &= \int \frac{u+1}{u}\,du\\ &= \int \left[1 + \frac{1}{u}\right] du && \text{divide}\\ &= u + \ln|u| + C && \text{constant and logarithmic rules}\\ &= x - 1 + \ln|x-1| + C && \text{substitute } x-1 \text{ for } u\end{aligned}$$

## EXAMPLE 5.2.5

Find $\displaystyle\int \frac{3x + 6}{\sqrt{2x^2 + 8x + 3}}\, dx.$

### Solution

This time, our guidelines suggest substituting for the quantity inside the radical in the denominator; that is,

$$u = 2x^2 + 8x + 3 \qquad du = (4x + 8)\, dx$$

At first glance, it may seem that this substitution fails, since $du = (4x + 8)\, dx$ appears quite different from the term $(3x + 6)\, dx$ in the integral. However, note that

$$\begin{aligned}(3x + 6)\, dx &= 3(x + 2)\, dx = \frac{3}{4}(4)[(x + 2)\, dx] \\ &= \frac{3}{4}[(4x + 8)\, dx] = \frac{3}{4}\, du\end{aligned}$$

Substituting, we find that

$$\begin{aligned}\int \frac{3x + 6}{\sqrt{2x^2 + 8x + 3}}\, dx &= \int \frac{1}{\sqrt{2x^2 + 8x + 3}}[(3x + 6)\, dx] \\ &= \int \frac{1}{\sqrt{u}}\left(\frac{3}{4}\, du\right) = \frac{3}{4}\int u^{-1/2}\, du \\ &= \frac{3}{4}\left(\frac{u^{1/2}}{1/2}\right) + C = \frac{3}{2}\sqrt{u} + C \\ &= \frac{3}{2}\sqrt{2x^2 + 8x + 3} + C \qquad \text{substitute } u = 2x^2 + 8x + 3\end{aligned}$$

## EXAMPLE 5.2.6

Find $\displaystyle\int \frac{(\ln x)^2}{x}\, dx.$

### Solution

Because

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

the integrand

$$\frac{(\ln x)^2}{x} = (\ln x)^2\left(\frac{1}{x}\right)$$

is a product in which one factor $\frac{1}{x}$ is the derivative of an expression $\ln x$ that appears in the other factor. This suggests substituting $u = \ln x$ with $du = \frac{1}{x}\,dx$ so that

$$\begin{aligned}\int \frac{(\ln x)^2}{x}\,dx &= \int (\ln x)^2\left(\frac{1}{x}\,dx\right)\\ &= \int u^2\,du = \frac{1}{3}u^3 + C\\ &= \frac{1}{3}(\ln x)^3 + C \qquad \text{substitute } \ln x \text{ for } u\end{aligned}$$

Sometimes an integral "looks" like it should be evaluated using a substitution but closer examination reveals a more direct approach. Consider Example 5.2.7.

## EXAMPLE 5.2.7

Find $\int e^{5x+2}\,dx$.

### Solution

You can certainly handle this integral using the substitution

$$u = 5x + 2 \qquad du = 5\,dx$$

but it is not really necessary since $e^{5x+2} = e^{5x}e^2$, and $e^2$ is a constant. Thus,

$$\begin{aligned}\int e^{5x+2}\,dx &= \int e^{5x}e^2\,dx\\ &= e^2\int e^{5x}\,dx \qquad \text{factor constant } e^2 \text{ outside integral}\\ &= e^2\left[\frac{e^{5x}}{5}\right] + C \qquad \text{exponential rule}\\ &= \frac{1}{5}e^{5x+2} + C \qquad \text{since } e^2e^{5x} = e^{5x+2}\end{aligned}$$

In Example 5.2.7, we used algebra to put the integrand into a form where substitution was not necessary. In Examples 5.2.8 and 5.2.9, we use algebra as a first step, before making a substitution.

## EXAMPLE 5.2.8

Find $\int \frac{x^2 + 3x + 5}{x + 1}\,dx$.

### Solution

There is no easy way to approach this integral as it stands (remember, there is no "quotient rule" for integration). However, suppose we simply divide the denominator into the numerator:

$$\begin{array}{r|l} & \quad x + 2 \\ \hline x + 1 & x^2 + 3x + 5 \\ & \underline{-x(x + 1)} \\ & \qquad 2x + 5 \\ & \qquad \underline{-2(x + 1)} \\ & \qquad\qquad 3 \end{array}$$

that is,

$$\frac{x^2 + 3x + 5}{x + 1} = x + 2 + \frac{3}{x + 1}$$

We can integrate $x + 2$ directly using the power rule. For the term $\dfrac{3}{x + 1}$, we use the substitution $u = x + 1$; $du = dx$:

$$\begin{aligned} \int \frac{x^2 + 3x + 5}{x + 1}\,dx &= \int \left[x + 2 + \frac{3}{x + 1}\right] dx \\ &= \int x\,dx + \int 2\,dx + \int \frac{3}{u}\,du && \begin{array}{l} u = x + 1 \\ du = dx \end{array} \\ &= \frac{1}{2}x^2 + 2x + 3\ln|u| + C \\ &= \frac{1}{2}x^2 + 2x + 3\ln|x + 1| + C && \text{substitute } x + 1 \text{ for } u \end{aligned}$$

## EXAMPLE 5.2.9

Find $\displaystyle\int \frac{1}{1 + e^{-x}}\,dx$.

### Solution

You may try to substitute $w = 1 + e^{-x}$. However, this is a dead end because $dw = -e^{-x}\,dx$ but there is no $e^{-x}$ term in the numerator of the integrand. Instead, note that

$$\begin{aligned} \frac{1}{1 + e^{-x}} &= \frac{1}{1 + \dfrac{1}{e^x}} = \frac{1}{\dfrac{e^x + 1}{e^x}} \\ &= \frac{e^x}{e^x + 1} \end{aligned}$$

Now, if you substitute $u = e^x + 1$ with $du = e^x\,dx$ into the given integral, you get

$$\begin{aligned}\int \frac{1}{1+e^{-x}}\,dx &= \int \frac{e^x}{e^x+1}\,dx = \int \frac{1}{e^x+1}(e^x\,dx)\\ &= \int \frac{1}{u}\,du\\ &= \ln|u| + C\\ &= \ln|e^x+1| + C \qquad \text{substitute } e^x + 1 \text{ for } u\end{aligned}$$

## When Substitution Fails

The method of substitution does not always succeed. In Example 5.2.10, we consider an integral very similar to the one in Example 5.2.3 but just enough different so no substitution will work.

### EXAMPLE 5.2.10

Evaluate $\int x^4 e^{x^4+2}\,dx$.

**Just-In-Time REVIEW**

Notice that if $u = x^4 + 2$, then $x^4 = u - 2$, so

$$x = (u-2)^{1/4} = \sqrt[4]{u-2}$$

#### Solution

The natural substitution is $u = x^4 + 2$, as in Example 5.2.3. As before, you find $du = 4x^3\,dx$, so $x^3\,dx = \frac{1}{4}\,du$, but this integrand involves $x^4$, not $x^3$. The "extra" factor of $x$ satisfies $x = \sqrt[4]{u-2}$, so when the substitution is made, you have

$$\int x^4 e^{x^4+2}\,dx = \int x e^{x^4+2}(x^3\,dx) = \int \sqrt[4]{u-2}\,e^u\,(\tfrac{1}{4}du)$$

which is hardly an improvement on the original integral! Try a few other possible substitutions (say, $u = x^2$ or $u = x^3$) to convince yourself that nothing works.

## An Application Involving Substitution

### EXAMPLE 5.2.11

The price $p$ (dollars) of each unit of a particular commodity is estimated to be changing at the rate

$$\frac{dp}{dx} = \frac{-135x}{\sqrt{9+x^2}}$$

where $x$ (hundred) units is the consumer demand (the number of units purchased at that price). Suppose 400 units ($x = 4$) are demanded when the price is \$30 per unit.

**a.** Find the demand function $p(x)$.

**b.** At what price will 300 units be demanded? At what price will no units be demanded?

**c.** How many units are demanded at a price of \$20 per unit?

## Solution

**a.** The price per unit $p(x)$ is found by integrating $p'(x)$ with respect to $x$. To perform this integration, use the substitution

$$u = 9 + x^2, \qquad du = 2x\,dx, \qquad x\,dx = \frac{1}{2}\,du$$

to get

$$\begin{aligned} p(x) &= \int \frac{-135x}{\sqrt{9+x^2}}\,dx = \int \frac{-135}{u^{1/2}}\left(\frac{1}{2}\right) du \\ &= \frac{-135}{2}\int u^{-1/2}\,du \\ &= \frac{-135}{2}\left(\frac{u^{1/2}}{1/2}\right) + C \\ &= -135\sqrt{9+x^2} + C \qquad \text{substitute } 9 + x^2 \text{ for } u \end{aligned}$$

Since $p = 30$ when $x = 4$, you find that

$$\begin{aligned} 30 &= -135\sqrt{9+4^2} + C \\ C &= 30 + 135\sqrt{25} = 705 \end{aligned}$$

so

$$p(x) = -135\sqrt{9+x^2} + 705$$

**b.** When 300 units are demanded, $x = 3$ and the corresponding price is

$$p(3) = -135\sqrt{9+3^2} + 705 = \$132.24 \text{ per unit}$$

No units are demanded when $x = 0$ and the corresponding price is

$$p(0) = -135\sqrt{9+0} + 705 = \$300 \text{ per unit}$$

**c.** To determine the number of units demanded at a unit price of \$20 per unit, you need to solve the equation

$$\begin{aligned} -135\sqrt{9+x^2} + 705 &= 20 \\ 135\sqrt{9+x^2} &= 685 \\ \sqrt{9+x^2} &= \frac{685}{135} \\ 9 + x^2 &\approx 25.75 \qquad \text{square both sides} \\ x^2 &\approx 16.75 \\ x &\approx 4.09 \end{aligned}$$

That is, roughly 409 units will be demanded when the price is \$20 per unit.

# EXERCISES 5.2

*In Exercises 1 and 2, fill in the table by specifying the substitution you would choose to find each of the four given integrals.*

**1.**

| Integral | Substitution $u$ |
|---|---|
| **a.** $\int (3x+4)^{5/2}\,dx$ | |
| **b.** $\int \frac{4}{3-x}\,dx$ | |
| **c.** $\int te^{2-t^2}\,dt$ | |
| **d.** $\int t(2+t^2)^3\,dt$ | |

**2.**

| Integral | Substitution $u$ |
|---|---|
| **a.** $\int \frac{3}{(2x-5)^4}\,dx$ | |
| **b.** $\int x^2e^{-x^3}\,dx$ | |
| **c.** $\int \frac{e^t}{e^t+1}\,dt$ | |
| **d.** $\int \frac{t+3}{\sqrt[3]{t^2+6t+5}}\,dt$ | |

*In Exercises 3 through 36, find the indicated integral and check your answer by differentiation.*

**3.** $\int (2x+6)^5\,dx$

**4.** $\int e^{5x+3}\,dx$

**5.** $\int \sqrt{4x-1}\,dx$

**6.** $\int \frac{1}{3x+5}\,dx$

**7.** $\int e^{1-x}\,dx$

**8.** $\int [(x-1)^5+3(x-1)^2+5]\,dx$

**9.** $\int xe^{x^2}\,dx$

**10.** $\int 2xe^{x^2-1}\,dx$

**11.** $\int t(t^2+1)^5\,dt$

**12.** $\int 3t\sqrt{t^2+8}\,dt$

**13.** $\int x^2(x^3+1)^{3/4}\,dx$

**14.** $\int x^5e^{1-x^6}\,dx$

**15.** $\int \frac{2y^4}{y^5+1}\,dy$

**16.** $\int \frac{y^2}{(y^3+5)^2}\,dy$

**17.** $\int (x+1)(x^2+2x+5)^{12}\,dx$

**18.** $\int (3x^2-1)e^{x^3-x}\,dx$

**19.** $\int \frac{3x^4+12x^3+6}{x^5+5x^4+10x+12}\,dx$

**20.** $\int \frac{10x^3-5x}{\sqrt{x^4-x^2+6}}\,dx$

**21.** $\int \frac{3u-3}{(u^2-2u+6)^2}\,du$

**22.** $\int \frac{6u-3}{4u^2-4u+1}\,du$

**23.** $\int \frac{\ln 5x}{x}\,dx$

**24.** $\int \frac{1}{x\ln x}\,dx$

**25.** $\displaystyle\int \frac{1}{x(\ln x)^2}\,dx$

**26.** $\displaystyle\int \frac{\ln x^2}{x}\,dx$

**27.** $\displaystyle\int \frac{2x\ln(x^2+1)}{x^2+1}\,dx$

**28.** $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

**29.** $\displaystyle\int \frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx$

**30.** $\displaystyle\int e^{-x}(1+e^{2x})\,dx$

**31.** $\displaystyle\int \frac{x}{2x+1}\,dx$

**32.** $\displaystyle\int \frac{t-1}{t+1}\,dt$

**33.** $\displaystyle\int x\sqrt{2x+1}\,dx$

**34.** $\displaystyle\int \frac{x}{\sqrt[3]{4-3x}}\,dx$

**35.** $\displaystyle\int \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx$

[*Hint:* Let $u=\sqrt{x}+1$.]

**36.** $\displaystyle\int \frac{1}{x^2}\left(\frac{1}{x}-1\right)^{2/3}dx$

$\left[\textit{Hint: } \text{Let } u=\dfrac{1}{x}-1.\right]$

*In Exercises 37 through 42, solve the given initial value problem for $y = f(x)$.*

**37.** $\dfrac{dy}{dx}=(3-2x)^2$ where $y=0$ when $x=0$

**38.** $\dfrac{dy}{dx}=\sqrt{4x+5}$ where $y=3$ when $x=1$

**39.** $\dfrac{dy}{dx}=\dfrac{1}{x+1}$ where $y=1$ when $x=0$

**40.** $\dfrac{dy}{dx}=e^{2-x}$ where $y=0$ when $x=2$

**41.** $\dfrac{dy}{dx}=\dfrac{x+2}{x^2+4x+5}$ where $y=3$ when $x=-1$

**42.** $\dfrac{dy}{dx}=\dfrac{\ln\sqrt{x}}{x}$ where $y=2$ when $x=1$

*In Exercises 43 through 46, the slope $f'(x)$ at each point $(x, y)$ on a curve $y = f(x)$ is given, along with a particular point $(a, b)$ on the curve. Use this information to find $f(x)$.*

**43.** $f'(x)=(1-2x)^{3/2}$; $(0, 0)$

**44.** $f'(x)=x\sqrt{x^2+5}$; $(2, 10)$

**45.** $f'(x)=xe^{4-x^2}$; $(-2, 1)$

**46.** $f'(x)=\dfrac{2x}{1+3x^2}$; $(0, 5)$

*In Exercises 47 through 50, the velocity $v(t) = x'(t)$ at time $t$ of an object moving along the $x$ axis is given, along with the initial position $x(0)$ of the object. In each case, find:*

*(a) The position $x(t)$ at time $t$.*
*(b) The position of the object at time $t = 4$.*
*(c) The time when the object is at $x = 3$.*

**47.** $x'(t)=-2(3t+1)^{1/2}$; $x(0)=4$

**48.** $x'(t)=\dfrac{-1}{1+0.5t}$; $x(0)=5$

**49.** $x'(t)=\dfrac{1}{\sqrt{2t+1}}$; $x(0)=0$

**50.** $x'(t)=\dfrac{-2t}{(1+t^2)^{3/2}}$; $x(0)=4$

**51.** **MARGINAL COST** At a certain factory, the marginal cost is $3(q-4)^2$ dollars per unit when the level of production is $q$ units.

**a.** Express the total production cost in terms of the overhead (the cost of producing 0 units) and the number of units produced.

**b.** What is the cost of producing 14 units if the overhead is \$436?

**52.** **DEPRECIATION** The resale value of a certain industrial machine decreases at a rate that depends on its age. When the machine is $t$ years old, the rate at which its value is changing is $-960e^{-t/5}$ dollars per year.

**a.** Express the value of the machine in terms of its age and initial value.

**b.** If the machine was originally worth \$5,200, how much will it be worth when it is 10 years old?

**53.** **TREE GROWTH** A tree has been transplanted and after $x$ years is growing at the rate of $1+\dfrac{1}{(x+1)^2}$ meters per year. After 2 years, it has reached a height of 5 meters. How tall was it when it was transplanted?

**54. RETAIL PRICES** In a certain section of the country, it is estimated that $t$ weeks from now, the price of chicken will be increasing at the rate of $p'(t) = 3\sqrt{t+1}$ cents per kilogram per week. If chicken currently costs \$2.30 per kilogram, what will it cost 8 weeks from now?

**55. REVENUE** The marginal revenue from the sale of $x$ units of a particular commodity is estimated to be

$$R'(x) = 50 + 3.5xe^{-0.01x^2} \quad \text{dollars per unit}$$

where $R(x)$ is revenue in dollars.

**a.** Find $R(x)$, assuming that $R(0) = 0$.

**b.** What revenue should be expected from the sale of 1,000 units?

**56. WATER POLLUTION** An oil spill in the ocean is roughly circular in shape, with radius $R(t)$ feet $t$ minutes after the spill begins. The radius is increasing at the rate

$$R'(t) = \frac{21}{0.07t + 5} \quad \text{ft/min}$$

**a.** Find an expression for the radius $R(t)$, assuming that $R = 0$ when $t = 0$.

**b.** What is the area $A = \pi R^2$ of the spill after 1 hour?

**57. DRUG CONCENTRATION** The concentration $C(t)$ in milligrams per cubic centimeter (mg/cm$^3$) of a drug in a patient's bloodstream is 0.5 mg/cm$^3$ immediately after an injection and $t$ minutes later is decreasing at the rate

$$C'(t) = \frac{-0.01e^{0.01t}}{(e^{0.01t} + 1)^2} \quad \text{mg/cm}^3 \text{ per minute}$$

A new injection is given when the concentration drops below 0.05 mg/cm$^3$.

**a.** Find an expression for $C(t)$.

**b.** What is the concentration after 1 hour? After 3 hours?

**c.** Use the graphing utility of your calculator with **TRACE** and **ZOOM** to determine how much time passes before the next injection is given.

**58. LAND VALUE** It is estimated that $x$ years from now, the value $V(x)$ of an acre of farmland will be increasing at the rate of

$$V'(x) = \frac{0.4x^3}{\sqrt{0.2x^4 + 8{,}000}}$$

dollars per year. The land is currently worth \$500 per acre.

**a.** Find $V(x)$.

**b.** How much will the land be worth in 10 years?

**c.** Use the graphing utility of your calculator with **TRACE** and **ZOOM** to determine how long it will take for the land to be worth \$1,000 per acre.

**59. AIR POLLUTION** In a certain suburb of Los Angeles, the level of ozone $L(t)$ at 7:00 A.M. is 0.25 parts per million (ppm). A 12-hour weather forecast predicts that the ozone level $t$ hours later will be changing at the rate of

$$L'(t) = \frac{0.24 - 0.03t}{\sqrt{36 + 16t - t^2}}$$

parts per million per hour (ppm/hr).

**a.** Express the ozone level $L(t)$ as a function of $t$. When does the peak ozone level occur? What is the peak level?

**b.** Use the graphing utility of your calculator to sketch the graph of $L(t)$ and use **TRACE** and **ZOOM** to answer the questions in part (a). Then determine at what other time the ozone level will be the same as it is at 11:00 A.M.

**60. SUPPLY** The owner of a fast-food chain determines that if $x$ thousand units of a new meal item are supplied, then the marginal price at that level of supply is given by

$$p'(x) = \frac{x}{(x+3)^2} \quad \text{dollars per meal}$$

where $p(x)$ is the price (in dollars) per unit at which all $x$ meal units will be sold. Currently, 5,000 units are being supplied at a price of \$2.20 per unit.

**a.** Find the supply (price) function $p(x)$.

**b.** If 10,000 meal units are supplied to restaurants in the chain, what unit price should be charged so that all the units will be sold?

**61. DEMAND** The manager of a shoe store determines that the price $p$ (dollars) for each pair of a popular brand of sports sneakers is changing at the rate of

$$p'(x) = \frac{-300x}{(x^2 + 9)^{3/2}}$$

when $x$ (hundred) pairs are demanded by consumers. When the price is \$75 per pair, 400 pairs ($x = 4$) are demanded by consumers.

**a.** Find the demand (price) function $p(x)$.
**b.** At what price will 500 pairs of sneakers be demanded? At what price will no sneakers be demanded?
**c.** How many pairs will be demanded at a price of \$90 per pair?

**62.** **SUPPLY** The price $p$ (dollars per unit) of a particular commodity is increasing at thc rate

$$p'(x) = \frac{20x}{(7 - x)^2}$$

when $x$ hundred units of the commodity are supplied to the market. The manufacturer supplies 200 units ($x = 2$) when the price is \$2 per unit.
**a.** Find the supply firnction $p(x)$.
**b.** What price corresponds to a supply of 500 units?

**63.** **MARGINAL PROFIT** A company determines that the marginal revenue from the production of $x$ units is $R'(x) = 7 - 3x - 4x^2$ hundred dollars per unit, and the corresponding marginal cost is $C'(x) = 5 + 2x$ hundred dollars per unit. By how much does the profit change when the level of production is raised from 5 to 9 units?

**64.** **MARGINAL PROFIT** Repeat Exercise 63 for marginal revenue $R'(x) = \dfrac{11 - x}{\sqrt{14 - x}}$ and for the marginal cost $C'(x) = 2 + x + x^2$.

**65.** Find $\int x^{1/3}(x^{2/3} + 1)^{3/2}\,dx$. [*Hint:* Substitute $u = x^{2/3} + 1$ and use $x^{2/3} = u - 1$.]

**66.** Find $\int x^3(4 - x^2)^{-1/2}\,dx$. [*Hint:* Substitute $u = 4 - x^2$ and use the fact that $x^2 = 4 - u$.]

**67.** Find $\int \dfrac{e^{2x}}{1 + e^x}\,dx$. [*Hint:* Let $u = 1 + e^x$.]

**68.** Find $\int e^{-x}(1 + e^x)^2\,dx$. [*Hint:* Is it better to set $u = 1 + e^x$ or $u = e^x$? Or is it better to not even use the method of substitution?]

## SECTION 5.3 The Definite Integral and the Fundamental Theorem of Calculus

Suppose a real estate agent wants to evaluate an unimproved parcel of land that is 100 feet wide and is bounded by streets on three sides and by a stream on the fourth side. The agent determines that if a coordinate system is set up as shown in Figure 5.2, the stream can be described by the curve $y = x^3 + 1$, where $x$ and $y$ are measured in hundreds of feet. If the area of the parcel is $A$ square feet and the agent estimates its land is worth \$12 per square foot, then the total value of the parcel is $12A$ dollars. If the parcel were rectangular in shape or triangular or even trapezoidal, its area $A$ could be found by substituting into a well-known formula, but the upper boundary of the parcel is curved, so how can the agent find the area and hence the total value of the parcel?

Our goal in this section is to show how area under a curve, such as the area $A$ in our real estate example, can be expressed as a limit of a sum of terms called a **definite integral.** We will then introduce a result called the **fundamental theorem of calculus** that allows us to compute *definite* integrals and thus find area and other quantities by using the *indefinite* integration (antidifferentiation) methods of Sections 5.1 and 5.2. In Example 5.3.3, we will illustrate this procedure by expressing the area $A$ in our real estate example as a definite integral and evaluating it using the fundamental theorem of calculus.

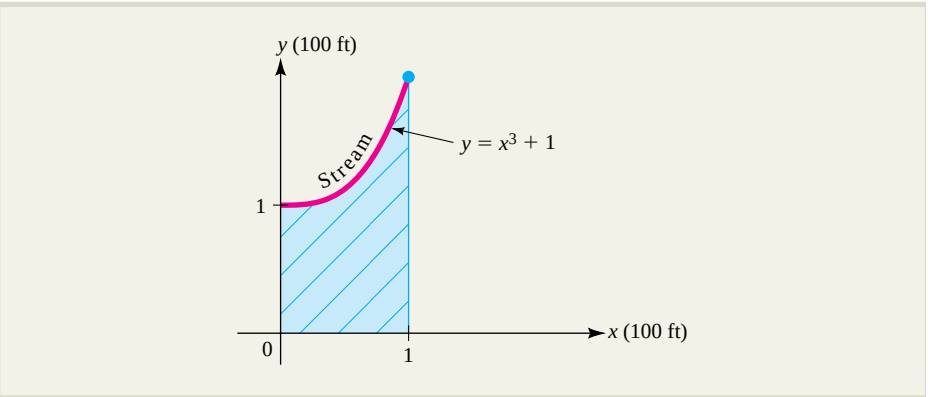



**FIGURE 5.2** Determining land value by finding the area under a curve.

## Area as the Limit of a Sum

Consider the area of the region under the curve $y = f(x)$ over an interval $a \le x \le b$, where $f(x) \ge 0$ and $f$ is continuous, as illustrated in Figure 5.3. To find this area, we will follow a useful general policy:

> *When faced with something you don't know how to handle, try to relate it to something you do know how to handle.*

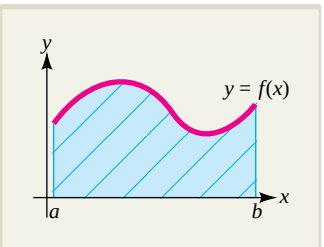



**FIGURE 5.3** The region under the curve $y = f(x)$ over the interval $a \le x \le b$.

In this particular case, we may not know the area under the given curve, but we do know how to find the area of a rectangle. Thus, we proceed by subdividing the region into a number of rectangular regions and then approximate the area $A$ under the curve $y = f(x)$ by adding the areas of the approximating rectangles.

To be more specific, begin the approximation by dividing the interval $a \le x \le b$ into $n$ equal subintervals, each of length $\Delta x = \dfrac{b - a^2}{n}$, and let $x_j$ denote the left endpoint of the $j$th subinterval, for $j = 1, 2, \ldots, n$. Then draw $n$ rectangles such that the $j$th rectangle has the $j$th subinterval as its base and $f(x_j)$ as its height. The approximation scheme is illustrated in Figure 5.4.

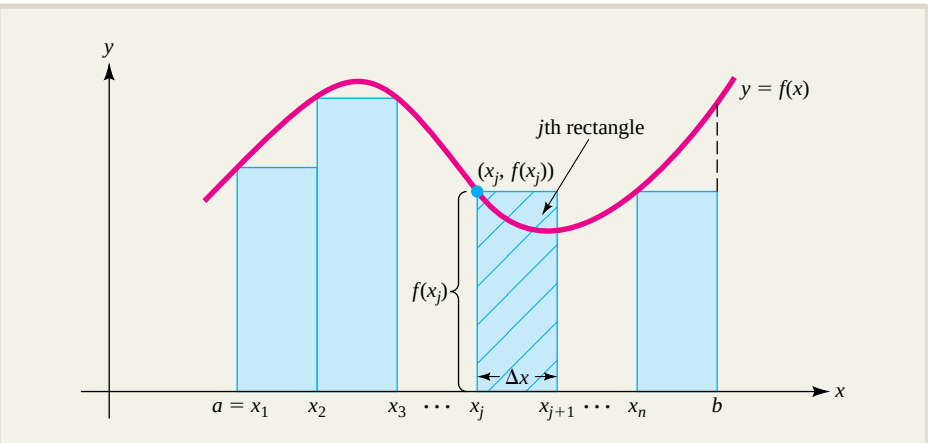



**FIGURE 5.4** An approximation of area under a curve by rectangles.

The area of the $j$th rectangle is $f(x_j)\,\Delta x$ and approximates the area under the curve above the subinterval $x_j \le x \le x_{j+1}$. The sum of the areas of all $n$ rectangles is

$$\begin{aligned} S_n &= f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \\ &= [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x \end{aligned}$$

which approximates the total area $A$ under the curve.

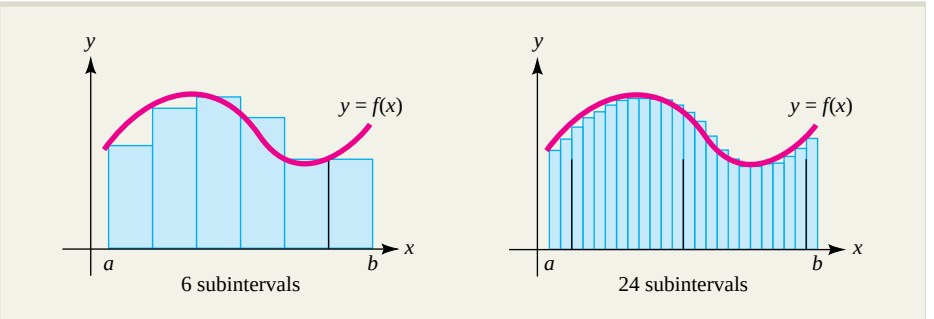



**FIGURE 5.5** The approximation improves as the number of subintervals increases.

As the number of subintervals $n$ increases, the approximating sum $S_n$ gets closer and closer to what we intuitively think of as the area under the curve, as illustrated in Figure 5.5. Therefore, it is reasonable to define the actual area $A$ under the curve as the limit of the sums. To summarize:

**Area Under a Curve** ■ Let $f(x)$ be continuous and satisfy $f(x) \ge 0$ on the interval $a \le x \le b$. Then the region under the curve $y = f(x)$ over the interval $a \le x \le b$ has area

$$A = \lim_{n \to +\infty} [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x$$

where $x_j$ is the left endpoint of the $j$th subinterval if the interval $a \le x \le b$ is divided into $n$ equal parts, each of length $\Delta x = \dfrac{b-a}{n}$.

**NOTE** At this point, you may ask, "Why use the left endpoint of the subintervals rather than, say, the right endpoint or even the midpoint?" The answer is that there is no reason we can't use those other points to compute the height of our approximating rectangles. In fact, the interval $a \le x \le b$ can be subdivided arbitrarily and arbitrary points chosen in each subinterval, and the result will still be the same. However, proving this equivalence is difficult, well beyond the scope of this text. ■

Here is an example in which area is computed as the limit of a sum and then checked using a geometric formula.

## EXAMPLE 5.3.1

Let $R$ be the region under the graph of $f(x) = 2x + 1$ over the interval $1 \le x \le 3$, as shown in Figure 5.6a. Compute the area of $R$ as the limit of a sum.

## Solution

The region $R$ is shown in Figure 5.6 with six approximating rectangles, each of width $\Delta x = \frac{3-1}{6} = \frac{1}{3}$. The left endpoints in the partition of $1 \le x \le 3$ are $x_1 = 1$, $x_2 = 1 + \frac{1}{3} = \frac{4}{3}$, and similarly, $x_3 = \frac{5}{3}$, $x_4 = 2$, $x_5 = \frac{7}{3}$, and $x_6 = \frac{8}{3}$. The corresponding values of $f(x) = 2x + 1$ are given in the following table:

| $x_j$ | 1 | $\frac{4}{3}$ | $\frac{5}{3}$ | 2 | $\frac{7}{3}$ | $\frac{8}{3}$ |
|---|---|---|---|---|---|---|
| $f(x_j) = 2x_j + 1$ | 3 | $\frac{11}{3}$ | $\frac{13}{3}$ | 5 | $\frac{17}{3}$ | $\frac{19}{3}$ |

Thus, the area $A$ of the region $R$ is approximated by the sum

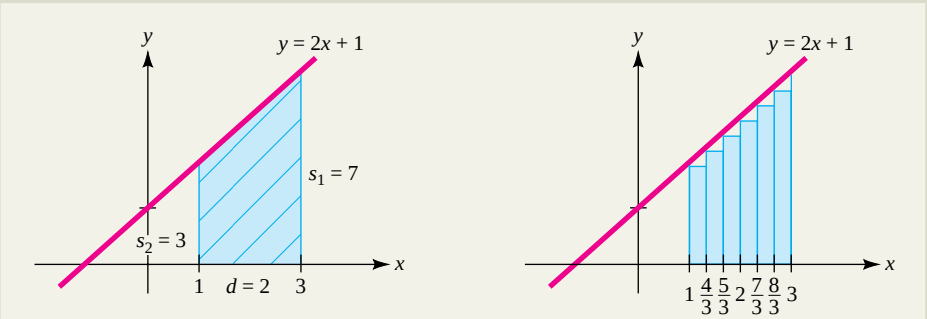
$$S = \left(3 + \frac{11}{3} + \frac{13}{3} + 5 + \frac{17}{3} + \frac{19}{3}\right)\left(\frac{1}{3}\right) = \frac{28}{3} \approx 9.333$$

### *Just-In-Time* REVIEW

A trapezoid is a four-sided polygon with at least two parallel sides. Its area is

$$A = \frac{1}{2}(s_1 + s_2)\,h$$

where $s_1$ and $s_2$ are the lengths of the two parallel sides and $h$ is the distance between them.




(a) The region $R$ under the line $y = 2x + 1$ over $1 \le x \le 3$ (a trapezoid)

(b) The region $R$ subdivided by six approximating rectangles

**FIGURE 5.6** Approximating the area under a line with rectangles.

If you continue to subdivide the region $R$ using more and more rectangles, the corresponding approximating sums $S_n$ approach the actual area $A$ of the region. The sum we have already computed for $n = 6$ is listed in the following table, along with those for $n = 10$, 20, 50, 100, and 500. (If you have access to a computer or a programmable calculator, see if you can write a program for generating any such sum for given $n$.)

| Number of rectangles $n$ | 6 | 10 | 20 | 50 | 100 | 500 |
|---|---|---|---|---|---|---|
| Approximating sum $S_n$ | 9.333 | 9.600 | 9.800 | 9.920 | 9.960 | 9.992 |

The numbers on the bottom line of this table seem to be approaching 10 as $n$ gets larger and larger. Thus, it is reasonable to conjecture that the region $R$ has area

$$A = \lim_{n \to +\infty} S_n = 10$$

Notice in Figure 5.6a that the region $R$ is a trapezoid of width $d = 3 - 1 = 2$ with parallel sides of lengths

$$s_1 = 2(3) + 1 = 7 \qquad \text{and} \qquad s_2 = 2(1) + 1 = 3$$

Such a trapezoid has area

$$A = \frac{1}{2}(s_1 + s_2)d = \frac{1}{2}(7 + 3)(2) = 10$$

the same result we just obtained using the limit of a sum procedure.

## The Definite Integral

Area is just one of many quantities that can be expressed as the limit of a sum. To handle all such cases, including those for which $f(x) \geq 0$ is *not* required and left endpoints are not used, we require the terminology and notation introduced in the following definition.

**EXPLORE!**

Place into Y1 the function $f(x) = -x^2 + 4x - 3$ and view its graph using the window [0, 4.7]1 by [−0.5, 1.5]0.5. Visually estimate the area under the curve from $x = 2$ to $x = 3$, using triangles or rectangles. Now use the numerical integration feature of your graphing calculator (**CALC** key, option 7). How far off were you and why?

**The Definite Integral** ■ Let $f(x)$ be a function that is continuous on the interval $a \leq x \leq b$. Subdivide the interval $a \leq x \leq b$ into $n$ equal parts, each of width $\Delta x = \dfrac{b-a}{n}$, and choose a number $x_k$ from the $k$th subinterval for $k = 1, 2, \ldots, n$. Form the sum

$$[f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x$$

called a **Riemann sum.**

Then the **definite integral** of $f$ on the interval $a \leq x \leq b$, denoted by $\int_a^b f(x)\,dx$, is the limit of the Riemann sum as $n \to +\infty$; that is,

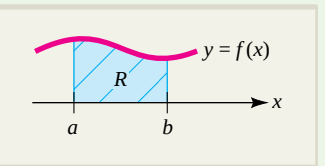
$$\int_a^b f(x)\,dx = \lim_{n \to +\infty} [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x$$

The function $f(x)$ is called the **integrand,** and the numbers $a$ and $b$ are called the **lower and upper limits of integration,** respectively. The process of finding a definite integral is called **definite integration.**

Surprisingly, the fact that $f(x)$ is continuous on $a \leq x \leq b$ turns out to be enough to guarantee that the limit used to define the definite integral $\int_a^b f(x)\,dx$ exists and is the same regardless of how the subinterval representatives $x_k$ are chosen.

The symbol $\int_a^b f(x)\,dx$ used for the definite integral is essentially the same as the symbol $\int f(x)\,dx$ for the indefinite integral, even though the definite integral is a specific number while the indefinite integral is a family of functions, the antiderivatives of $f$. You will soon see that these two apparently very different concepts are intimately related. Here is a compact form for the definition of area using the integral notation.

**Area as a Definite Integral** ■ If $f(x)$ is continuous and $f(x) \geq 0$ on the interval $a \leq x \leq b$, then the region $R$ under the curve $y = f(x)$ over the interval $a \leq x \leq b$ has area $A$ given by the definite integral $A = \int_a^b f(x)\,dx$.



## The Fundamental Theorem of Calculus

If computing the limit of a sum were the only way of evaluating a definite integral, the integration process probably would be little more than a mathematical novelty. Fortunately, there is an easier way of performing this computation, thanks to this remarkable result connecting the definite integral to antidifferentiation.

**The Fundamental Theorem of Calculus** ■ If the function $f(x)$ is continuous on the interval $a \le x \le b$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$ on $a \le x \le b$.

A special case of the fundamental theorem of calculus is verified at the end of this section. When applying the fundamental theorem, we use the notation

$$F(x)\Big|_a^b = F(b) - F(a)$$

Thus,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

**NOTE** You may wonder how the fundamental theorem of calculus can promise that if $F(x)$ is *any* antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

To see why this is true, suppose $G(x)$ is another such antiderivative. Then $G(x) = F(x) + C$ for some constant $C$, so $F(x) = G(x) - C$ and

$$\begin{aligned}\int_a^b f(x)\,dx &= F(b) - F(a)\\ &= [G(b) - C] - [G(a) - C]\\ &= G(b) - G(a)\end{aligned}$$

since the $C$'s cancel. Thus, the valuation is the same regardless of which antiderivative is used. ■

In Example 5.3.2, we demonstrate the computational value of the fundamental theorem of calculus by using it to compute the same area we estimated as the limit of a sum in Example 5.3.1.

### EXAMPLE 5.3.2

Use the fundamental theorem of calculus to find the area of the region under the line $y = 2x + 1$ over the interval $1 \le x \le 3$.

### Solution

Since $f(x) = 2x + 1$ satisfies $f(x) \geq 0$ on the interval $1 \leq x \leq 3$, the area is given by the definite integral $A = \int_1^3 (2x + 1)\,dx$. Since an antiderivative of $f(x) = 2x + 1$ is $F(x) = x^2 + x$, the fundamental theorem of calculus tells us that

$$A = \int_1^3 (2x + 1)\,dx = x^2 + x\Big|_1^3$$
$$= [(3)^2 + (3)] - [(1)^2 + (1)] = 10$$

as estimated in Example 5.3.1.

**EXPLORE!**

Refer to Example 5.3.3. Use the numerical integration feature of your calculator to confirm numerically that

$$\int_0^1 (x^3 + 1)\,dx = 1.25$$

## EXAMPLE 5.3.3

Find the area of the parcel of land described in the introduction to this section; that is, the area under the curve $y = x^3 + 1$ over the interval $0 \leq x \leq 1$, where $x$ and $y$ are in hundreds of feet. If the land in the parcel is appraised at \$12 per square foot, what is the total value of the parcel?

### Solution

The area of the parcel is given by the definite integral

$$A = \int_0^1 (x^3 + 1)\,dx$$

Since an antiderivative of $f(x) = x^3 + 1$ is $F(x) = \frac{1}{4}x^4 + x$, the fundamental theorem of calculus tells us that

$$A = \int_0^1 (x^3 + 1)\,dx = \frac{1}{4}x^4 + x\Big|_0^1$$
$$= \left[\frac{1}{4}(1)^4 + 1\right] - \left[\frac{1}{4}(0)^4 + 0\right] = \frac{5}{4}$$

Because $x$ and $y$ are measured in hundreds of feet, the total area is

$$\frac{5}{4} \times 100 \times 100 = 12{,}500 \text{ ft}^2$$

and since the land in the parcel is worth \$12 per square foot, the total value of the parcel is

$$V = (\$12/\text{ft}^3)(12{,}500 \text{ ft}^2) = \$150{,}000$$

## EXAMPLE 5.3.4

Evaluate the definite integral $\int_0^1 (e^{-x} + \sqrt{x})\,dx$.

**Just-In-Time REVIEW**

When the fundamental theorem of calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

is used to evaluate a definite integral, remember to compute ***both*** $F(b)$ and $F(a)$, even when $a = 0$.

### Solution

An antiderivative of $f(x) = e^{-x} + \sqrt{x}$ is $F(x) = -e^{-x} + \frac{2}{3}x^{3/2}$, so the definite integral is

$$\begin{aligned}\int_0^1 (e^{-x} + \sqrt{x})\,dx &= \left(-e^{-x} + \frac{2}{3}x^{3/2}\right)\Big|_0^1 \\ &= \left[-e^{-1} + \frac{2}{3}(1)^{3/2}\right] - \left[-e^0 + \frac{2}{3}(0)\right] \\ &= \frac{5}{3} - \frac{1}{e} \approx 1.299\end{aligned}$$

Our definition of the definite integral was motivated by computing area, which is a nonnegative quantity. However, since the definition does not require $f(x) \geq 0$, it is quite possible for a definite integral to be negative, as illustrated in Example 5.3.5.

### EXAMPLE 5.3.5

Evaluate $\displaystyle\int_1^4 \left(\frac{1}{x} - x^2\right) dx$.

### Solution

An antiderivative of $f(x) = \frac{1}{x} - x^2$ is $F(x) = \ln|x| - \frac{1}{3}x^3$, so we have

$$\begin{aligned}\int_1^4 \left(\frac{1}{x} - x^2\right) dx &= \left(\ln|x| - \frac{1}{3}x^3\right)\Big|_1^4 \\ &= \left[\ln 4 - \frac{1}{3}(4)^3\right] - \left[\ln 1 - \frac{1}{3}(1)^3\right] \\ &= \ln 4 - 21 \approx -19.6137\end{aligned}$$

## Integration Rules

This list of rules can be used to simplify the computation of definite integrals.

### Rules for Definite Integrals

Let $f$ and $g$ be any functions continuous on $a \leq x \leq b$. Then,

1. **Constant multiple rule:** $\displaystyle\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$ for constant $k$
2. **Sum rule:** $\displaystyle\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. **Difference rule:** $\displaystyle\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$
4. $\displaystyle\int_a^a f(x)\,dx = 0$
5. $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
6. **Subdivision rule:** $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Rules 4 and 5 are really special cases of the definition of the definite integral. The first three rules can be proved by using the fundamental theorem of calculus along with an analogous rule for indefinite integrals. For instance, to verify the constant multiple rule, suppose $F(x)$ is an antiderivative of $f(x)$. Then, according to the constant multiple rule for indefinite integrals, $kF(x)$ is an antiderivative of $kf(x)$ and the fundamental theorem of calculus tells us that

$$\begin{aligned}\int_a^b kf(x)\,dx &= kF(x)\Big|_a^b \\ &= kF(b) - kF(a) = k[F(b) - F(a)] \\ &= k\int_a^b f(x)\,dx\end{aligned}$$

You are asked to verify the sum rule using similar reasoning in Exercise 70.

In the case where $f(x) \geq 0$ on the interval $a \leq x \leq b$, the subdivision rule is a geometric reflection of the fact that the area under the curve $y = f(x)$ over the interval $a \leq x \leq b$ is the sum of the areas under $y = f(x)$ over the subintervals $a \leq x \leq c$ and $c \leq x \leq b$, as illustrated in Figure 5.7. However, it is important to remember that the subdivision rule is true even if $f(x)$ does *not* satisfy $f(x) \geq 0$ on $a \leq x \leq b$.

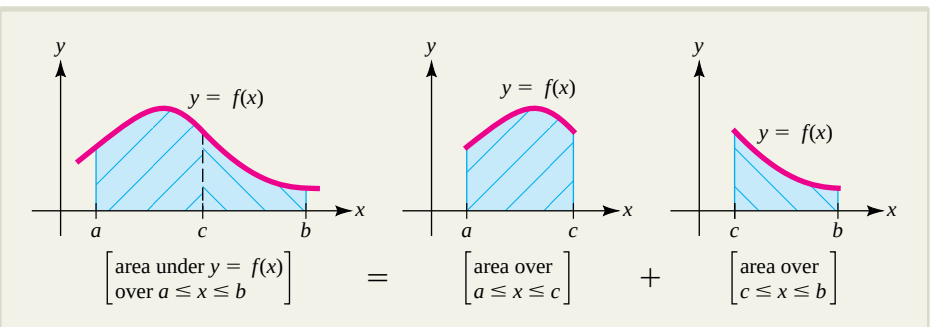



**FIGURE 5.7** The subdivision rule for definite integrals [case where $f(x) \geq 0$].

## EXAMPLE 5.3.6

Let $f(x)$ and $g(x)$ be functions that are continuous on the interval $-2 \leq x \leq 5$ and that satisfy

$$\int_{-2}^{5} f(x)\,dx = 3 \qquad \int_{-2}^{5} g(x)\,dx = -4 \qquad \int_{3}^{5} f(x)\,dx = 7$$

Use this information to evaluate each of these definite integrals:

**a.** $\displaystyle\int_{-2}^{5} [2f(x) - 3g(x)]\,dx$ **b.** $\displaystyle\int_{-2}^{3} f(x)\,dx$

### Solution

**a.** By combining the difference rule and constant multiple rule and substituting the given information, we find that

$$\begin{aligned}\int_{-2}^{5}[2f(x)-3g(x)]\,dx &= \int_{-2}^{5}2f(x)\,dx-\int_{-2}^{5}3g(x)\,dx && \text{difference rule}\\ &= 2\int_{-2}^{5}f(x)\,dx-3\int_{-2}^{5}g(x)\,dx && \text{constant multiple rule}\\ &= 2(3)-3(-4)=18 && \text{substitute given information}\end{aligned}$$

**b.** According to the subdivision rule

$$\int_{-2}^{5}f(x)\,dx=\int_{-2}^{3}f(x)\,dx+\int_{3}^{5}f(x)\,dx$$

Solving this equation for the required integral $\int_{-2}^{3}f(x)\,dx$ and substituting the given information, we obtain

$$\begin{aligned}\int_{-2}^{3}f(x)\,dx &= \int_{-2}^{5}f(x)\,dx-\int_{3}^{5}f(x)\,dx\\ &= 3-7=-4\end{aligned}$$

## Substituting in a Definite Integral

When using a substitution $u=g(x)$ to evaluate a definite integral $\int_a^b f(x)\,dx$, you can proceed in either of these two ways:

1. Use the substitution to find an antiderivative $F(x)$ for $f(x)$ and then evaluate the definite integral using the fundamental theorem of calculus.
2. Use the substitution to express the integrand and $dx$ in terms of $u$ and $du$ and to replace the original limits of integration, $a$ and $b$, with transformed limits $c=g(a)$ and $d=g(b)$. The original integral can then be evaluated by applying the fundamental theorem of calculus to the transformed definite integral.

These procedures are illustrated in Examples 5.3.7 and 5.3.8.

**Just-In-Time REVIEW**

Only one member of the family of antiderivatives of $f(x)$ is needed for evaluating $\int_a^b f(x)\,dx$ by the fundamental theorem of calculus. Therefore, the "$+C$" may be left out of intermediate integrations.

### EXAMPLE 5.3.7

Evaluate $\int_0^1 8x(x^2+1)^3\,dx$.

### Solution

The integrand is a product in which one factor $8x$ is a constant multiple of the derivative of an expression $x^2+1$ that appears in the other factor. This suggests that you let $u=x^2+1$. Then $du=2x\,dx$, and so

$$\int 8x(x^2+1)^3\,dx=\int 4u^3\,du=u^4$$

The limits of integration, 0 and 1, refer to the variable $x$ and not to $u$. You can, therefore, proceed in one of two ways. Either you can rewrite the antiderivative in terms of $x$, or you can find the values of $u$ that correspond to $x=0$ and $x=1$.

If you choose the first alternative, you find that

$$\int 8x(x^2+1)^3\,dx = u^4 = (x^2+1)^4$$

and so
$$\int_0^1 8x(x^2+1)^3\,dx = (x^2+1)^4\Big|_0^1 = 16 - 1 = 15$$

If you choose the second alternative, use the fact that $u = x^2 + 1$ to conclude that $u = 1$ when $x = 0$ and $u = 2$ when $x = 1$. Hence,

$$\int_0^1 8x(x^2+1)^3\,dx = \int_1^2 4u^3\,du = u^4\Big|_1^2 = 16 - 1 = 15$$

**EXPLORE!**

Refer to Example 5.3.8. Use a graphing utility with the window [0, 3]1 by [−4, 1]1 to graph $f(x) = \ln \frac{x}{x}$. Explain in terms of area why the integral of $f(x)$ over $\frac{1}{4} \le x \le 2$ is negative.

## EXAMPLE 5.3.8

Evaluate $\displaystyle\int_{1/4}^{2} \left(\frac{\ln x}{x}\right) dx$.

### Solution

Let $u = \ln x$, so $du = \frac{1}{x}\,dx$. Then

$$\int \frac{\ln x}{x}\,dx = \int \ln x\left(\frac{1}{x}\,dx\right) = \int u\,du$$
$$= \frac{1}{2}u^2 = \frac{1}{2}(\ln x)^2$$

Thus,

$$\int_{1/4}^{2} \frac{\ln x}{x}\,dx = \left[\frac{1}{2}(\ln x)^2\right]\Big|_{1/4}^{2} = \frac{1}{2}(\ln 2)^2 - \frac{1}{2}\left(\ln\frac{1}{4}\right)^2$$
$$= -\frac{3}{2}(\ln 2)^2 \approx -0.721$$

Alternatively, use the substitution $u = \ln x$ to transform the limits of integration:

$$\text{when } x = \frac{1}{4}, \text{ then } u = \ln\frac{1}{4}$$
$$\text{when } x = 2, \text{ then } u = \ln 2$$

Substituting, we find

$$\int_{1/4}^{2} \frac{\ln x}{x}\,dx = \int_{\ln 1/4}^{\ln 2} u\,du = \frac{1}{2}u^2\Big|_{\ln 1/4}^{\ln 2}$$
$$= \frac{1}{2}(\ln 2)^2 - \frac{1}{2}\left(\ln\frac{1}{4}\right)^2 \approx -0.721$$

## Net Change

In certain applications, we are given the rate of change $Q'(x)$ of a quantity $Q(x)$ and required to compute the **net change** $Q(b) - Q(a)$ in $Q(x)$ as $x$ varies from $x = a$ to $x = b$. We did this in Section 5.1 by solving initial value problems (recall Examples

5.1.5 through 5.1.8). However, since $Q(x)$ is an antiderivative of $Q'(x)$, the fundamental theorem of calculus allows us to compute net change by the following definite integration formula.

**Net Change** ■ If $Q'(x)$ is continuous on the interval $a \le x \le b$, then the **net change** in $Q(x)$ as $x$ varies from $x = a$ to $x = b$ is given by

$$Q(b) - Q(a) = \int_a^b Q'(x)\,dx$$

Here are two examples involving net change.

## EXAMPLE 5.3.9

At a certain factory, the marginal cost is $3(q-4)^2$ dollars per unit when the level of production is $q$ units. By how much will the total manufacturing cost increase if the level of production is raised from 6 units to 10 units?

### Solution

Let $C(q)$ denote the total cost of producing $q$ units. Then the marginal cost is the derivative $\frac{dC}{dq} = 3(q-4)^2$, and the increase in cost if production is raised from 6 units to 10 units is given by the definite integral

$$\begin{aligned} C(10) - C(6) &= \int_6^{10} \frac{dC}{dq}\,dq \\ &= \int_6^{10} 3(q-4)^2\,dq = (q-4)^3\Big|_6^{10} \\ &= (10-4)^3 - (6-4)^3 \\ &= \$208 \end{aligned}$$

## EXAMPLE 5.3.10

A protein with mass $m$ (grams) disintegrates into amino acids at a rate given by

$$\frac{dm}{dt} = \frac{-30}{(t+3)^2} \quad \text{g/hr}$$

What is the net change in mass of the protein during the first 2 hours?

### Solution

The net change is given by the definite integral

$$m(2) - m(0) = \int_0^2 \frac{dm}{dt}\,dt = \int_0^2 \frac{-30}{(t+3)^2}\,dt$$

If we substitute $u = t + 3$, $du = dt$, and change the limits of integration accordingly ($t = 0$ becomes $u = 3$ and $t = 2$ becomes $u = 5$), we find

$$\begin{aligned} m(2) - m(0) &= \int_0^2 \frac{-30}{(t+3)^2}\,dt = \int_3^5 -30u^{-2}\,du \\ &= -30\left(\frac{u^{-1}}{-1}\right)\Bigg|_3^5 = 30\left[\frac{1}{5} - \frac{1}{3}\right] \\ &= -4 \end{aligned}$$

Thus, the mass of the protein has a net decrease of 4 g over the first 2 hours.

## Area Justification of the Fundamental Theorem of Calculus

We close this section with a justification of the fundamental theorem of calculus for the case where $f(x) \geq 0$. In this case, the definite integral $\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$ over the interval $[a, b]$. For fixed $x$ between $a$ and $b$, let $A(x)$ denote the area under $y = f(x)$ over the interval $[a, x]$. Then the difference quotient of $A(x)$ is

$$\frac{A(x+h) - A(x)}{h}$$

and the expression $A(x + h) - A(x)$ in the numerator is just the area under the curve $y = f(x)$ between $x$ and $x + h$. If $h$ is small, this area is approximately the same as the area of the rectangle with height $f(x)$ and width $h$ as indicated in Figure 5.8. That is,

$$A(x+h) - A(x) \approx f(x)h$$

or, equivalently,

$$\frac{A(x+h) - A(x)}{h} \approx f(x)$$

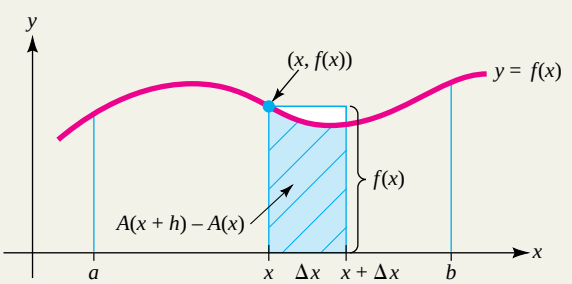



**FIGURE 5.8** The area $A(x + h) - A(x)$.

As $h$ approaches 0, the error in the approximation approaches 0, and it follows that

$$\lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = f(x)$$

But by the definition of the derivative,

$$\lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = A'(x)$$

so that

$$A'(x) = f(x)$$

In other words, $A(x)$ is an antiderivative of $f(x)$.

Suppose $F(x)$ is any other antiderivative of $f(x)$. Then, according to the fundamental property of antiderivatives (Section 5.1), we have

$$A(x) = F(x) + C$$

for some constant $C$ and all $x$ in the interval $a \le x \le b$. Since $A(x)$ represents the area under $y = f(x)$ between $a$ and $x$, it is certainly true that $A(a)$, the area between $a$ and $a$, is 0, so that

$$A(a) = 0 = F(a) + C$$

and $C = -F(a)$. The area under $y = f(x)$ between $x = a$ and $x = b$ is $A(b)$, which satisfies

$$A(b) = F(b) + C = F(b) - F(a)$$

Finally, since the area under $y = f(x)$ above $a \le x \le b$ is also given by the definite integral $\int_a^b f(x)\,dx$, it follows that

$$\int_a^b f(x)\,dx = A(b) = F(b) - F(a)$$

as claimed in the fundamental theorem of calculus.

## EXERCISES ■ 5.3

*In Exercises 1 through 30, evaluate the given definite integral using the fundamental theorem of calculus.*

**1.** $\displaystyle\int_{-1}^{2} 5\,dx$

**2.** $\displaystyle\int_{-2}^{1} \pi\,dx$

**3.** $\displaystyle\int_{0}^{5} (3x + 2)\,dx$

**4.** $\displaystyle\int_{1}^{4} (5 - 2t)\,dt$

**5.** $\displaystyle\int_{-1}^{1} 3t^4\,dt$

**6.** $\displaystyle\int_{1}^{4} 2\sqrt{u}\,du$

**7.** $\displaystyle\int_{-1}^{1} (2u^{1/3} - u^{2/3})\,du$

**8.** $\displaystyle\int_{4}^{9} x^{-3/2}\,dx$

**9.** $\displaystyle\int_{0}^{1} e^{-x}(4 - e^{x})\,dx$

**10.** $\displaystyle\int_{-1}^{1} \left(\frac{1}{e^x} - \frac{1}{e^{-x}}\right) dx$

**11.** $\displaystyle\int_{0}^{1} (x^4 + 3x^3 + 1)\,dx$

**12.** $\displaystyle\int_{-1}^{0} (-3x^5 - 3x^2 + 2x + 5)\,dx$

**13.** $\displaystyle\int_{2}^{5} (2 + 2t + 3t^2)\,dt$

**14.** $\displaystyle\int_1^9 \left(\sqrt{t} - \frac{4}{\sqrt{t}}\right) dt$

**15.** $\displaystyle\int_1^3 \left(1 + \frac{1}{x} + \frac{1}{x^2}\right) dx$

**16.** $\displaystyle\int_0^{\ln 2} (e^t - e^{-t})\, dt$

**17.** $\displaystyle\int_{-3}^{-1} \frac{t+1}{t^3}\, dt$

**18.** $\displaystyle\int_1^6 x^2(x-1)\, dx$

**19.** $\displaystyle\int_1^2 (2x-4)^4\, dx$

**20.** $\displaystyle\int_{-3}^0 (2x+6)^4\, dx$

**21.** $\displaystyle\int_0^4 \frac{1}{\sqrt{6t+1}}\, dt$

**22.** $\displaystyle\int_1^2 \frac{x^2}{(x^3+1)^2}\, dx$

**23.** $\displaystyle\int_0^1 (x^3+x)\sqrt{x^4+2x^2+1}\, dx$

**24.** $\displaystyle\int_0^1 \frac{6t}{t^2+1}\, dt$

**25.** $\displaystyle\int_2^{e+1} \frac{x}{x-1}\, dx$

**26.** $\displaystyle\int_1^2 (t+1)(t-2)^6\, dt$

**27.** $\displaystyle\int_1^{e^2} \frac{(\ln x)^2}{x}\, dx$

**28.** $\displaystyle\int_e^{e^2} \frac{1}{x \ln x}\, dx$

**29.** $\displaystyle\int_{1/3}^{1/2} \frac{e^{1/x}}{x^2}\, dx$

**30.** $\displaystyle\int_1^4 \frac{(\sqrt{x}-1)^{3/2}}{\sqrt{x}}\, dx$

*In Exercises 31 through 38, f(x) and g(x) are functions that are continuous on the interval $-3 \le x \le 2$ and satisfy*

$$\int_{-3}^2 f(x)\, dx = 5 \qquad \int_{-3}^2 g(x)\, dx = -2 \qquad \int_{-3}^1 f(x)\, dx = 0 \qquad \int_{-3}^1 g(x)\, dx = 4$$

*In each case, use this information along with rules for definite integrals to evaluate the indicated integral.*

**31.** $\displaystyle\int_{-3}^2 [-2f(x) + 5g(x)]\, dx$

**32.** $\displaystyle\int_{-3}^1 [4f(x) - 3g(x)]\, dx$

**33.** $\displaystyle\int_4^4 g(x)\, dx$

**34.** $\displaystyle\int_2^{-3} f(x)\, dx$

**35.** $\displaystyle\int_1^2 f(x)\, dx$

**36.** $\displaystyle\int_1^2 g(x)\, dx$

**37.** $\displaystyle\int_1^2 [3f(x) + 2g(x)]\, dx$

**38.** $\displaystyle\int_{-3}^1 [2f(x) + 3g(x)]\, dx$

*In Exercises 39 through 46, find the area of the region R that lies under the given curve $y = f(x)$ over the indicated interval $a \le x \le b$.*

**39.** Under $y = x^4$, over $-1 \le x \le 2$

**40.** Under $y = \sqrt{x}\,(x+1)$, over $0 \le x \le 4$

**41.** Under $y = (3x+4)^{1/2}$, over $0 \le x \le 4$

**42.** Under $y = \dfrac{3}{\sqrt{9-2x}}$, over $-8 \le x \le 0$

**43.** Under $y = e^{2x}$, over $0 \le x \le \ln 3$

**44.** Under $y = xe^{-x^2}$, over $0 \le x \le 3$

**45.** Under $y = \dfrac{3}{5-2x}$, over $-2 \le x \le 1$

**46.** Under $y = \dfrac{3}{x}$, over $1 \le x \le e^2$

**47.** **LAND VALUES** It is estimated that $t$ years from now the value of a certain parcel of land will be increasing at the rate of $V'(t)$ dollars per year. Find an expression for the amount by which the value of the land will increase during the next 5 years.

**48.** **ADMISSION TO EVENTS** The promoters of a county fair estimate that $t$ hours after the gates open at 9:00 A.M., visitors will be entering the fair at the rate of $N'(t)$ people per hour. Find an expression for the number of people who will enter the fair between 11:00 A.M. and 1:00 P.M.

**49.** **STORAGE COST** A retailer receives a shipment of 12,000 pounds of soybeans that will be used at a constant rate of 300 pounds per week. If the cost of storing the soybeans is 0.2 cent per pound per week, how much will the retailer have to pay in storage costs over the next 40 weeks?

**50.** **OIL PRODUCTION** A certain oil well that yields 400 barrels of crude oil a month will run dry in 2 years. The price of crude oil is currently \$95 per barrel and is expected to rise at a constant rate of 30 cents per barrel per month. If the oil is sold as soon as it is extracted from the ground, what will be the total future revenue from the well?

**51.** **AIR POLLUTION** An environmental study of a certain community suggests that $t$ years from now the level $L(t)$ of carbon monoxide in the air will be changing at the rate of $L'(t) = 0.1t + 0.1$ parts per million (ppm) per year. By how much will the pollution level change during the next 3 years?

**52.** **WATER POLLUTION** It is estimated that $t$ years from now the population of a certain lakeside community will be changing at the rate of $0.6t^2 + 0.2t + 0.5$ thousand people per year. Environmentalists have found that the level of pollution in the lake increases at the rate of approximately 5 units per 1,000 people. By how much will the pollution in the lake increase during the next 2 years?

**53.** **NET GROWTH OF POPULATION** A study indicates that $t$ months from now the population of a certain town will be growing at the rate of $P'(t) = 5 + 3t^{2/3}$ people per month. By how much will the population of the town increase over the next 8 months?

**54.** **MARGINAL COST** The marginal cost of producing a certain commodity is $C'(q) = 6q + 1$ dollars per unit when $q$ units are being produced.

**a.** What is the total cost of producing the first 10 units?

**b.** What is the cost of producing the *next* 10 units?

**55.** **FARMING** It is estimated that $t$ days from now a farmer's crop will be increasing at the rate of $0.3t^2 + 0.6t + 1$ bushels per day. By how much will the value of the crop increase during the next 5 days if the market price remains fixed at \$3 per bushel?

**56.** **SALES REVENUE** It is estimated that the demand for a manufacturer's product is increasing exponentially at the rate of 2% per year. If the current demand is 5,000 units per year and if the price remains fixed at \$400 per unit, how much revenue will the manufacturer receive from the sale of the product over the next 2 years?

**57.** **PRODUCTION** Bejax Corporation has set up a production line to manufacture a new type of cellular telephone. The rate of production of the telephones is

$$\frac{dP}{dt} = 1{,}500\left(2 - \frac{t}{2t+5}\right) \quad \text{units/month}$$

How many telephones are produced during the third month?

**58.** **PRODUCTION** The output of a factory is changing at the rate

$$Q'(t) = 2t^3 - 3t^2 + 10t + 3 \text{ units/hour}$$

where $t$ is the number of hours after the morning shift begins at 8 A.M. How many units are produced between 10 A.M. and noon?

**59.** **INVESTMENT** An investment portfolio changes value at the rate

$$V'(t) = 12e^{-0.05t}(e^{0.3t} - 3)$$

where $V$ is in thousands of dollars and $t$ is the number of years after 2004. By how much does the value of the protfolio change between the years:

**a.** 2004 and 2008

**b.** 2008 and 2010

**60.** **ADVERTISING** An advertising agency begins a campaign to promote a new product and determines that $t$ days later, the number of people $N(t)$ who

have heard about the product is changing at a rate given by

$$N'(t) = 5t^2 - \frac{0.04t}{t^2 + 3} \quad \text{people per day}$$

How many people learn about the product during the first week? During the second week?

**61.** **CONCENTRATION OF DRUG** The concentration of a drug in a patient's bloodstream $t$ hours after an injection is decreasing at the rate

$$C'(t) = \frac{-0.33t}{\sqrt{0.02t^2 + 10}} \quad \text{mg/cm}^3 \text{ per hour}$$

By how much does the concentration change over the first 4 hours after the injection?

**62.** **ENDANGERED SPECIES** A study conducted by an environmental group in the year 2000 determined that $t$ years later, the population of a certain endangered bird species will be decreasing at the rate of $P'(t) = -0.75t\sqrt{10 - 0.2t}$ individuals per year. By how much is the population expected to change during the decade 2000–2010?

**63.** **DEPRECIATION** The resale value of a certain industrial machine decreases over a 10-year period at a rate that changes with time. When the machine is $x$ years old, the rate at which its value is changing is $220(x - 10)$ dollars per year. By how much does the machine depreciate during the second year?

**64.** **WATER CONSUMPTION** The city manager of Paloma Linda estimates that water is being consumed by his community at the rate of $C'(t) = 10 + 0.3e^{0.03t}$ billion gallons per year, where $C(t)$ is the water consumption $t$ years after the year 2000. How much water will be consumed by the community during the decade 2000–2010?

**65.** **CHANGE IN BIOMASS** A protein with mass $m$ (grams) disintegrates into amino acids at a rate given by

$$\frac{dm}{dt} = \frac{-2}{t + 1} \quad \text{g/hr}$$

How much more protein is there after 2 hours than after 5 hours?

**66.** **CHANGE IN BIOMASS** Answer the question in Exercise 65 if the rate of disintegration is given by

$$\frac{dm}{dt} = -(0.1t + e^{0.1t})$$

**67.** **RATE OF LEARNING** In a learning experiment, subjects are given a series of facts to memorize, and it is determined that $t$ minutes after the experiment begins, the average subject is learning at the rate

$$L'(t) = \frac{4}{\sqrt{t + 1}} \quad \text{facts per minute}$$

where $L(t)$ is the total number of facts memorized by time $t$. About how many facts does the typical subject learn during the second 5 minutes (between $t = 5$ and $t = 10$)?

**68.** **DISTANCE AND VELOCITY** A driver, traveling at a constant speed of 45 mph, decides to speed up in such a way that her velocity $t$ hours later is $v(t) = 45 + 12t$ mph. How far does she travel in the first 2 hours?

**69.** **PROJECTILE MOTION** A ball is thrown upward from the top of a building, and $t$ seconds later has velocity $v(t) = -32t + 80$ ft/sec. What is the difference in the ball's position after 3 seconds?

**70.** Verify the sum rule for definite integrals; that is, if $f(x)$ and $g(x)$ are continuous on the interval $a \le x \le b$, then

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

**71.** You have seen that the definite integral can be used to compute the area under a curve, but the "area as an integral" formula works both ways.

**a.** Compute $\int_0^1 \sqrt{1 - x^2}\,dx$. [*Hint:* Note that the integral is part of the area under the circle $x^2 + y^2 = 1$.]

**b.** Compute $\int_1^2 \sqrt{2x - x^2}\,dx$. [*Hint:* Describe the graph of $y = \sqrt{2x - x^2}$ and look for a geometric solution as in part (a).]

**72.** Given the function of $f(x) = 2\sqrt{x} + \dfrac{1}{x+1}$, approximate the value of the integral $\int_0^2 f(x)\,dx$ by completing these steps:

**a.** Find the numbers $x_1$, $x_2$, $x_3$, $x_4$, and $x_5$ that subdivide the interval $0 \le x \le 2$ into four equal subintervals. Use these numbers to form four rectangles that approximate the area under the curve $y = f(x)$ over $0 \le x \le 2$.

**b.** Estimate the value of the given integral by computing the sum of the areas of the four approximating rectangles in part (a).

**c.** Repeat steps (a) and (b) with eight subintervals instead of four.

## SECTION 5.4 Applying Definite Integration: Area Between Curves and Average Value

We have seen that area can be expressed as a special kind of limit of a sum called a definite integral and then computed by applying the fundamental theorem of calculus. This procedure, called **definite integration,** was introduced through area because area is easy to visualize, but there are many applications other than area in which the integration process plays an important role.

In this section, we extend the ideas introduced in Section 5.3 to find the area between two curves and the average value of a function. As part of our study of area between curves, we will examine an important socioeconomic device called a Lorentz curve, which is used to measure relative wealth within a society.

### Applying the Definite Integral

Intuitively, definite integration can be thought of as a process that "accumulates" an infinite number of small pieces of a quantity to obtain the total quantity. Here is a step-by-step description of how to use this process in applications.

**A Procedure for Using Definite Integration in Applications**

To use definite integration to "accumulate" a quantity $Q$ over an interval $a \le x \le b$, proceed as follows:

**Step 1.** Divide the interval $a \le x \le b$ into $n$ equal subintervals, each of length $\Delta x = \dfrac{b-a}{n}$. Choose a number $x_j$ from the $j$th subinterval, for $j = 1, 2, \ldots, n$.

**Step 2.** Approximate small parts of the quantity $Q$ by products of the form $f(x_j)\Delta x$, where $f(x)$ is an appropriate function that is continuous on $a \le x \le b$.

**Step 3.** Add the individual approximating products to estimate the total quantity $Q$ by the Riemann sum

$$[f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x$$

**Step 4.** Make the approximation in step 3 exact by taking the limit of the Riemann sum as $n \to +\infty$ to express $Q$ as a definite integral; that is,

$$Q = \lim_{n \to +\infty} [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x = \int_a^b f(x)\,dx$$

Then use the fundamental theorem of calculus to compute $\int_a^b f(x)\,dx$ and thus to obtain the required quantity $Q$.

**Just-In-Time REVIEW**

The summation notation is reviewed in Appendix A4 including examples. Note that there is nothing special about using "$j$" for the index in the notation. The most commonly used indices are $i$, $j$, and $k$.

**NOTATION:** We can use *summation notation* to represent the Riemann sums that occur when quantities are modeled using definite integration. Specifically, to describe the sum

$$a_1 + a_2 + \cdots + a_n$$

it suffices to specify the general term $a_j$ in the sum and to indicate that $n$ terms of this form are to be added, starting with the term where $j = 1$ and ending with $j = n$. For this purpose, it is customary to use the uppercase Greek letter sigma ($\Sigma$) and to write the sum as $\sum_{j=1}^{n} a_j$, that is,

$$\sum_{j=1}^{n} a_j = a_1 + a_2 + \cdots + a_n$$

In particular, the Riemann sum

$$[f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x$$

can be written in the compact form

$$\sum_{j=1}^{n} f(x_j)\Delta x$$

Thus, the limit statement

$$\lim_{n\to+\infty} [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x = \int_a^b f(x)\,dx$$

used to define the definite integral can be expressed as

$$\lim_{n\to+\infty} \sum_{j=1}^{n} f(x_j)\Delta x = \int_a^b f(x)\,dx$$

## Area Between Two Curves

In certain practical applications, you may find it useful to represent a quantity of interest in terms of area between two curves. First, suppose that $f$ and $g$ are continuous, nonnegative [that is, $f(x) \geq 0$ and $g(x) \geq 0$], and satisfy $f(x) \geq g(x)$ on the interval $a \leq x \leq b$, as shown in Figure 5.9a.

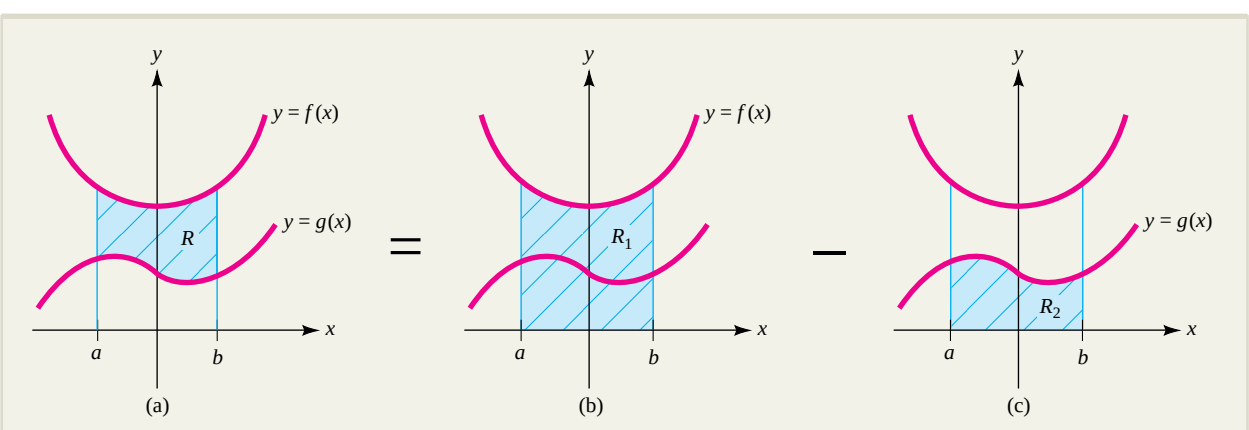



**FIGURE 5.9** Area of $R$ = area of $R_1$ − area of $R_2$.

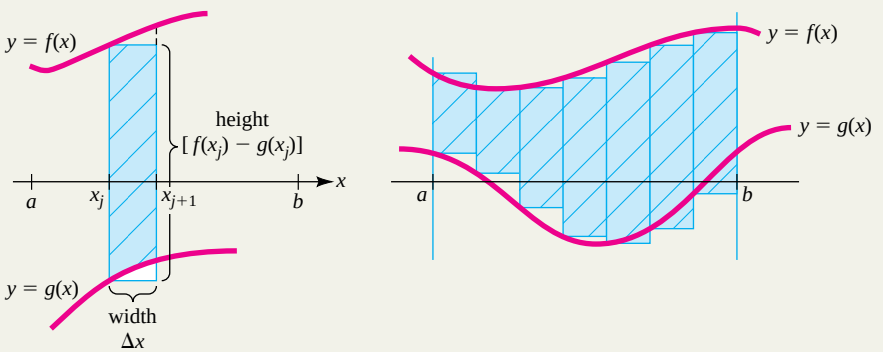
Then, to find the area of the region $R$ between the curves $y = f(x)$ and $y = g(x)$ over the interval $a \le x \le b$, we simply subtract the area under the lower curve $y = g(x)$ (Figure 5.9c) from the area under the upper curve $y = f(x)$ (Figure 5.9b), so that

$$\begin{aligned}\text{Area of } R &= [\text{area under } y = f(x)] - [\text{area under } y = g(x)]\\ &= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx\end{aligned}$$

This formula still applies whenever $f(x) \ge g(x)$ on the interval $a \le x \le b$, even when the curves $y = f(x)$ and $y = g(x)$ are not always both above the $x$ axis. We will show that this is true by using the procedure for applying definite integration described on page 414.

**Step 1.** Subdivide the interval $a \le x \le b$ into $n$ equal subintervals, each of width $\Delta x = \dfrac{b-a}{n}$. For $j = 1, 2, \ldots, n$, let $x_j$ be the left endpoint of the $j$th subinterval.

**Step 2.** Construct approximating rectangles of width $\Delta x$ and height $f(x_j) - g(x_j)$. This is possible since $f(x) \ge g(x)$ on $a \le x \le b$, which guarantees that the height is nonnegative; that is $f(x_j) - g(x_j) \ge 0$. For $j = 1, 2, \ldots, n$, the area $[f(x_j) - g(x_j)]\Delta x$ of the $j$th rectangle you have just constructed is approximately the same as the area between the two curves over the $j$th subinterval, as shown in Figure 5.10a.




(a) The area of the $j$th rectangle approximates the area between the curves on the $j$th subinterval.

(b) The total area between the curves is estimated by the sum of the areas of the approximating rectangles.

**FIGURE 5.10** Computing area between curves by definite integration.

**Step 3.** Add the individual approximating areas $[f(x_j) - g(x_j)]\Delta x$ to estimate the total area $A$ between the two curves over the interval $a \le x \le b$ by the Riemann sum

$$\begin{aligned}A &\approx [f(x_1) - g(x_1)]\Delta x + [f(x_2) - g(x_2)]\Delta x + \cdots + [f(x_n) - g(x_n)]\Delta x\\ &= \sum_{j=1}^{n} [f(x_j) - g(x_j)]\Delta x\end{aligned}$$

(See Figure 5.10b.)

**Step 4.** Make the approximation exact by taking the limit of the Riemann sum in step 3 as $n \to +\infty$ to express the total area $A$ between the curves as a definite integral; that is,

$$A = \lim_{n \to +\infty} \sum_{j=1}^{n} [f(x_j) - g(x_j)]\Delta x = \int_a^b [f(x) - g(x)]\, dx$$

To summarize:

**The Area Between Two Curves** ■ If $f(x)$ and $g(x)$ are continuous with $f(x) \geq g(x)$ on the interval $a \leq x \leq b$, then the area $A$ between the curves $y = f(x)$ and $y = g(x)$ over the interval is given by

$$A = \int_a^b [f(x) - g(x)]\, dx$$

## EXAMPLE 5.4.1

Find the area of the region $R$ enclosed by the curves $y = x^3$ and $y = x^2$.

### Solution

To find the points where the curves intersect, solve the equations simultaneously as follows:

$$\begin{aligned} x^3 &= x^2 \\ x^3 - x^2 &= 0 && \text{subtract } x^2 \text{ from both sides} \\ x^2(x-1) &= 0 && \text{factor out } x^2 \\ x &= 0,\ 1 && uv = 0 \text{ if and only if } u = 0 \text{ or } v = 0 \end{aligned}$$

The corresponding points (0, 0) and (1, 1) are the only points of intersection.

The region $R$ enclosed by the two curves is bounded above by $y = x^2$ and below by $y = x^3$, over the interval $0 \leq x \leq 1$ (Figure 5.11). The area of this region is given by the integral

$$\begin{aligned} A &= \int_0^1 (x^2 - x^3)\, dx = \frac{1}{3}x^3 - \frac{1}{4}x^4 \Big|_0^1 \\ &= \left[\frac{1}{3}(1)^3 - \frac{1}{4}(1)^4\right] - \left[\frac{1}{3}(0)^3 - \frac{1}{4}(0)^4\right] = \frac{1}{12} \end{aligned}$$

*Just-In-Time* **REVIEW**

Note that $x^2 \geq x^3$ for $0 \leq x \leq 1$. For example,

$$\left(\frac{1}{3}\right)^2 > \left(\frac{1}{3}\right)^3$$

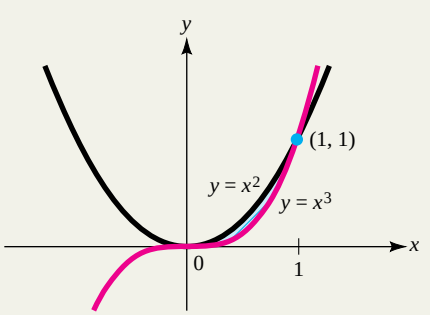



**FIGURE 5.11** The region enclosed by the curves $y = x^2$ and $y = x^3$.

In certain applications, you may need to find the area $A$ between the two curves $y = f(x)$ and $y = g(x)$ over an interval $a \le x \le b$, where $f(x) \ge g(x)$ for $a \le x \le c$ but $g(x) \ge f(x)$ for $c \le x \le b$. In this case, we have

$$A = \underbrace{\int_a^c [f(x) - g(x)]\,dx}_{f(x) \ge g(x) \text{ on } a \le x \le c} + \underbrace{\int_c^b [g(x) - f(x)]\,dx}_{g(x) \ge f(x) \text{ on } c \le x \le b}$$

Consider Example 5.4.2.

**EXPLORE!**

Refer to Example 5.4.2. Set Y1 = 4X and Y2 = $X^3 + 3X^2$ in the equation editor of your graphing calculator. Graph using the window [−6, 2]1 by [−25, 10]5. Determine the points of intersection of the two curves. Another view of the area between the two curves is to set Y3 = Y2 − Y1, deselect (turn off) Y1 and Y2, and graph using [−4.5, 1.5]0.5 by [−5, 15]5. Numerical integration can be applied to this difference function.

## EXAMPLE 5.4.2

Find the area of the region enclosed by the line $y = 4x$ and the curve $y = x^3 + 3x^2$.

### Solution

To find where the line and curve intersect, solve the equations simultaneously as follows:

$$\begin{aligned} x^3 + 3x^2 &= 4x \\ x^3 + 3x^2 - 4x &= 0 && \text{subtract } 4x \text{ from both sides} \\ x(x^2 + 3x - 4) &= 0 && \text{factor out } x \\ x(x-1)(x+4) &= 0 && \text{factor } x^2 + 3x - 4 \\ x &= 0,\ 1,\ -4 && uv = 0 \text{ if and only if } u = 0 \text{ or } v = 0 \end{aligned}$$

The corresponding points of intersection are (0, 0), (1, 4), and (−4, −16). The curve and the line are sketched in Figure 5.12.

Over the interval $-4 \le x \le 0$, the curve is above the line, so $x^3 + 3x^2 \ge 4x$, and the region enclosed by the curve and line has area

$$\begin{aligned} A_1 &= \int_{-4}^{0} [(x^3 + 3x^2) - 4x]\,dx = \frac{1}{4}x^4 + x^3 - 2x^2 \Big|_{-4}^{0} \\ &= \left[\frac{1}{4}(0)^4 + (0)^3 - 2(0)^2\right] - \left[\frac{1}{4}(-4)^4 + (-4)^3 - 2(-4)^2\right] = 32 \end{aligned}$$

Over the interval $0 \le x \le 1$, the line is above the curve and the enclosed region has area

$$\begin{aligned} A_2 &= \int_0^1 [4x - (x^3 + 3x^2)]\,dx = 2x^2 - \frac{1}{4}x^4 - x^3 \Big|_0^1 \\ &= \left[2(1)^2 - \frac{1}{4}(1)^4 - (1)^3\right] - \left[2(0)^2 - \frac{1}{4}(0)^4 - (0)^3\right] = \frac{3}{4} \end{aligned}$$

Therefore, the total area enclosed by the line and the curve is given by the sum

$$A = A_1 + A_2 = 32 + \frac{3}{4} = 32.75$$

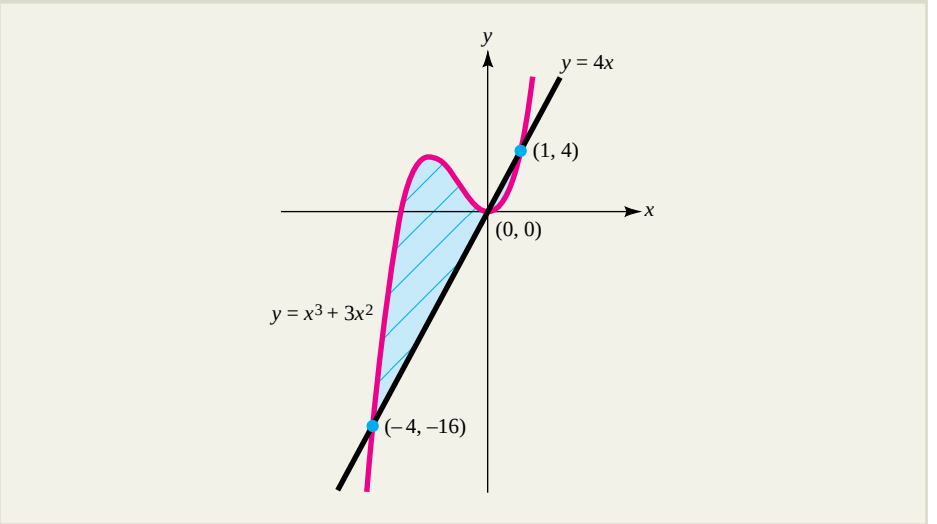



**FIGURE 5.12** The region enclosed by the line $y = 4x$ and the curve $y = x^3 + 3x^2$.

## Net Excess Profit

The area between curves can sometimes be used as a way of measuring the amount of a quantity that has been accumulated during a particular procedure. For instance, suppose that $t$ years from now, two investment plans will be generating profit $P_1(t)$ and $P_2(t)$, respectively, and that their respective rates of profitability, $P_1'(t)$ and $P_2'(t)$, are expected to satisfy $P_2'(t) \geq P_1'(t)$ for the next $N$ years; that is, over the time interval $0 \leq t \leq N$. Then $E(t) = P_2(t) - P_1(t)$ represents the **excess profit** of plan 2 over plan 1 at time $t$, and the **net excess profit** $\text{NE} = E(N) - E(0)$ over the time interval $0 \leq t \leq N$ is given by the definite integral

$$\text{NE} = E(N) - E(0) = \int_0^N E'(t)\,dt$$

$$= \int_0^N [P_2'(t) - P_1'(t)]\,dt \qquad \text{since } E'(t) = [P_2(t) - P_1(t)]' = P_2'(t) - P_1'(t)$$

This integral can be interpreted geometrically as the area between the rate of profitability curves $y = P_1'(t)$ and $y = P_2'(t)$ as shown in Figure 5.13. Example 5.4.3 illustrates the computation of net excess profit.

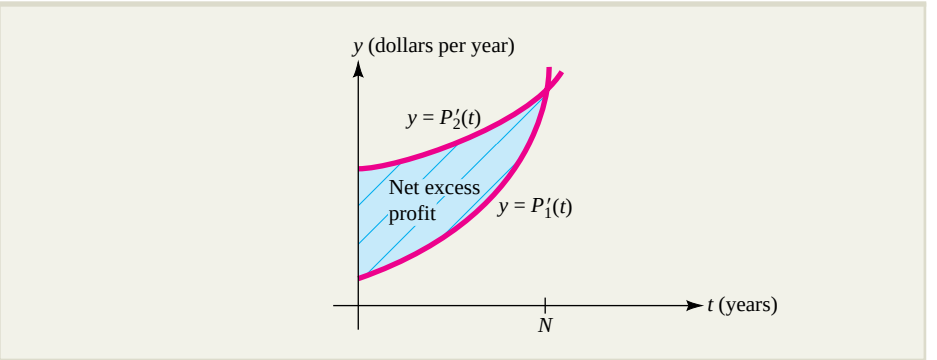



**FIGURE 5.13** Net excess profit as the area between rate of profitability curves.

## EXAMPLE 5.4.3

Suppose that $t$ years from now, one investment will be generating profit at the rate of $P_1'(t) = 50 + t^2$ hundred dollars per year, while a second investment will be generating profit at the rate of $P_2'(t) = 200 + 5t$ hundred dollars per year.

**a.** For how many years does the rate of profitability of the second investment exceed that of the first?

**b.** Compute the net excess profit for the time period determined in part (a). Interpret the net excess profit as an area.

### Solution

**a.** The rate of profitability of the second investment exceeds that of the first until

$$\begin{aligned} P_1'(t) &= P_2'(t) \\ 50 + t^2 &= 200 + 5t \\ t^2 - 5t - 150 &= 0 && \text{subtract } 200 + 5t \text{ from both sides} \\ (t - 15)(t + 10) &= 0 && \text{factor} \\ t &= 15,\ -10 && \text{since } uv = 0 \text{ if and only if } u = 0 \text{ or } v = 0 \\ t &= 15 \text{ years} && \text{reject the negative time } t = -10 \end{aligned}$$

**b.** The excess profit of plan 2 over plan 1 is $E(t) = P_2(t) - P_1(t)$, and the net excess profit NE over the time period $0 \le t \le 15$ determined in part (a) is given by the definite integral

$$\begin{aligned} \text{NE} = E(15) - E(0) &= \int_0^{15} E'(t)\,dt && \text{fundamental theorem of calculus} \\ &= \int_0^{15} [P_2'(t) - P_1'(t)]\,dt && \text{since } E(t) = P_2(t) - P_1(t) \\ &= \int_0^{15} [(200 + 5t) - (50 + t^2)]\,dt \\ &= \int_0^{15} [150 + 5t - t^2]\,dt && \text{combine terms} \\ &= \left[150t + 5\left(\frac{1}{2}t^2\right) - \left(\frac{1}{3}t^3\right)\right]\Bigg|_0^{15} \\ &= \left[150(15) + \frac{5}{2}(15)^2 - \frac{1}{3}(15)^3\right] - \left[150(0) + \frac{5}{2}(0)^2 - \frac{1}{3}(0)^3\right] \\ &= 1{,}687.50 \text{ hundred dollars} \end{aligned}$$

Thus, the net excess profit is \$168,750.

The graphs of the rate of profitability functions $P_1'(t)$ and $P_2'(t)$ are shown in Figure 5.14. The net excess profit

$$\text{NE} = \int_0^{15} [P_2'(t) - P_1'(t)]\,dt$$

can be interpreted as the area of the (shaded) region between the rate of profitability curves over the interval $0 \le t \le 15$.

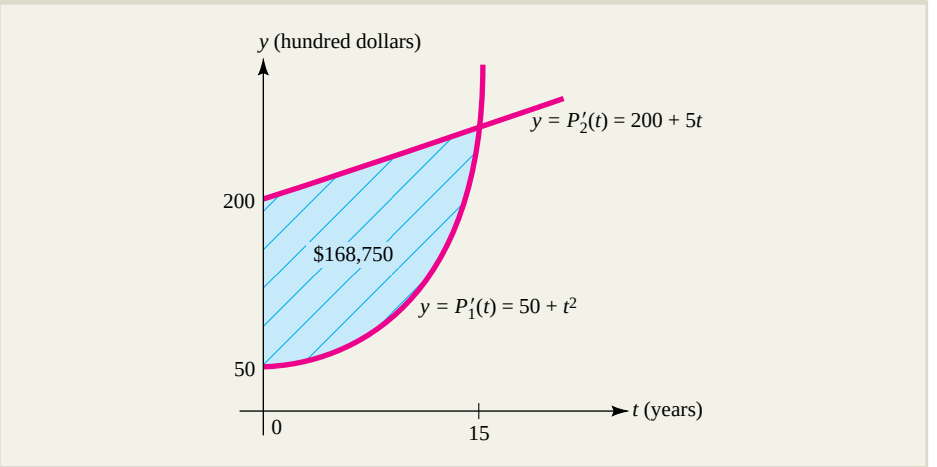



**FIGURE 5.14** Net excess profit for one investment plan over another.

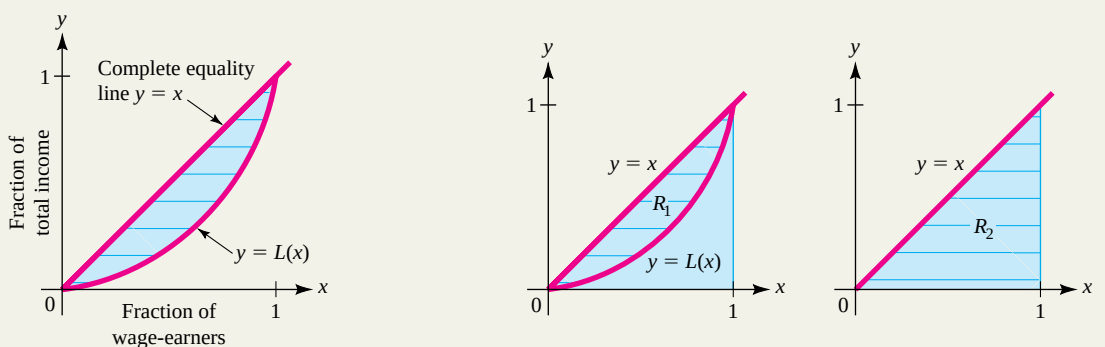
## Lorentz Curves

Area also plays an important role in the study of **Lorentz curves,** a device used by both economists and sociologists to measure the percentage of a society's wealth that is possessed by a given percentage of its people. To be more specific, the Lorentz curve for a particular society's economy is the graph of the function $L(x)$, which denotes the fraction of total annual national income earned by the lowest-paid $100x\%$ of the wage-earners in the society, for $0 \le x \le 1$. For instance, if the lowest-paid 30% of all wage-earners receive 23% of the society's total income, then $L(0.3) = 0.23$.

Note that $L(x)$ is an increasing function on the interval $0 \le x \le 1$ and has these properties:

1. $0 \le L(x) \le 1$ because $L(x)$ is a percentage
2. $L(0) = 0$ because no wages are earned when no wage-earners are employed
3. $L(1) = 1$ because 100% of wages are earned by 100% of the wage-earners
4. $L(x) \le x$ because the lowest-paid $100x\%$ of wage-earners cannot receive more than $100x\%$ of total income

A typical Lorentz curve is shown in Figure 5.15a.




(a) A Lorentz curve $y = L(x)$ lies below the complete equality line $y = x$

(b) The Gini index is the ratio $\text{GI} = \dfrac{\text{area of } R_1}{\text{area of } R_2}$

**FIGURE 5.15** A Lorentz curve $y = L(x)$ and its Gini index.

The line $y = x$ represents the ideal case corresponding to complete equality in the distribution of income (wage-earners with the lowest $100x\%$ of income receive $100x\%$ of the society's wealth). The closer a particular Lorentz curve is to this line, the more equitable the distribution of wealth in the corresponding society. We represent the total deviation of the actual distribution of wealth in the society from complete equality by the area of the region $R_1$ between the Lorentz curve $y = L(x)$ and the line $y = x$. The ratio of this area to the area of the region $R_2$ under the complete equality line $y = x$ over $0 \le x \le 1$ is used as a measure of the inequality in the distribution of wealth in the society. This ratio, called the **Gini index,** denoted GI (also called the **index of income inequality**), may be computed by the formula

$$\begin{aligned}
\text{GI} &= \frac{\text{area of } R_1}{\text{area of } R_2} = \frac{\text{area between } y = L(x) \text{ and } y = x}{\text{area under } y = x \text{ over } 0 \le x \le 1} \\
&= \frac{\displaystyle\int_0^1 [x - L(x)]\,dx}{\displaystyle\int_0^1 x\,dx} = \frac{\displaystyle\int_0^1 [x - L(x)]\,dx}{1/2} \\
&= 2\int_0^1 [x - L(x)]\,dx
\end{aligned}$$

(see Figure 5.15b). To summarize:

**Gini Index** ■ If $y = L(x)$ is the equation of a Lorentz curve, then the inequality in the corresponding distribution of wealth is measured by the *Gini index,* which is given by the formula

$$\text{Gini index} = 2\int_0^1 [x - L(x)]\,dx$$

The Gini index always lies between 0 and 1. An index of 0 corresponds to total equity in the distribution of income, while an index of 1 corresponds to total inequity (all income belongs to 0% of the population). The smaller the index, the more equitable the distribution of income, and the larger the index, the more the wealth is concentrated in only a few hands. Example 5.4.4 illustrates how Lorentz curves and the Gini index can be used to compare the relative equity of income distribution for two professions.

## EXAMPLE 5.4.4

A governmental agency determines that the Lorentz curves for the distribution of income for dentists and contractors in a certain state are given by the functions

$$L_1(x) = x^{1.7} \qquad \text{and} \qquad L_2(x) = 0.8x^2 + 0.2x$$

respectively. For which profession is the distribution of income more fairly distributed?

### Solution

The respective Gini indices are

$$G_1 = 2\int_0^1 (x - x^{1.7})\,dx = 2\left(\frac{x^2}{2} - \frac{x^{2.7}}{2.7}\right)\Bigg|_0^1 = 0.2593$$

and

$$G_2 = 2\int_0^1 [x - (0.8x^2 + 0.2x)]\,dx$$
$$= 2\left[-0.8\left(\frac{x^3}{3}\right) + 0.8\left(\frac{x^2}{2}\right)\right]\Bigg|_0^1 = 0.2667$$

Since the Gini index for dentists is smaller, it follows that in this state, the incomes of dentists are more evenly distributed than those of contractors.

Using the Gini index, we can see how the distribution of income in the United States compares to that in other countries. Table 5.1 lists the Gini index for selected industrial and developing nations. Note that with an index of 0.46, the distribution of income in the United States is about the same as that of Thailand, is less equitable than the United Kingdom, Germany, or Denmark, but much more equitable than Brazil or Panama. (Is there anything you know about the socio-political nature of these countries that would explain the difference in income equity?)

**TABLE 5.1** Gini Indices for Selected Countries

| Country | Gini Index |
|---|---|
| United States | 0.460 |
| Brazil | 0.601 |
| Canada | 0.315 |
| Denmark | 0.247 |
| Germany | 0.281 |
| Japan | 0.350 |
| South Africa | 0.584 |
| Panama | 0.568 |
| Thailand | 0.462 |
| United Kingdom | 0.326 |

SOURCE: David C. Colander, *Economics*, 4th ed., Boston: McGraw-Hill, 2001, p. 435.

## Average Value of a Function

As a second illustration of how definite integration can be used in applications, we will compute the **average value of a function,** which is of interest in a variety of situations. First, let us take a moment to clarify our thinking about what we mean by "average value." A teacher who wants to compute the average score on an examination simply adds all the individual scores and then divides by the number of students taking the exam, but how should one go about finding, say, the average pollution level in a city during the daytime hours? The difficulty is that since time is continuous, there are "too many" pollution levels to add up in the usual way, so how should we proceed?

Consider the general case in which we wish to find the average value of the function $f(x)$ over an interval $a \le x \le b$ on which $f$ is continuous. We begin by subdividing the interval $a \le x \le b$ into $n$ equal parts, each of length $\Delta x = \dfrac{b-a}{n}$. If $x_j$ is a number

**EXPLORE!**

Suppose you wish to calculate the average value of $f(x) = x^3 - 6x^2 + 10x - 1$ over the interval [1, 4]. Store $f(x)$ in Y1 and obtain its graph using the window [0, 4.7]1 by [−2, 8]1. Shade the region under the curve over the interval [1, 4] and compute its area $A$. Set Y2 equal to the constant function $\frac{A}{b-a} = \frac{A}{3}$. This is the average value. Plot Y2 and Y1 on the same screen. At what number(s) between 1 and 4 does $f(x)$ equal the average value?

taken from the $j$th subinterval for $j = 1, 2, \ldots, n$, then the average of the corresponding functional values $f(x_1), f(x_2), \ldots, f(x_n)$ is

$$\begin{aligned}
V_n &= \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \\
&= \frac{b-a}{b-a}\left[\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}\right] && \text{multiply and divide by } (b-a) \\
&= \frac{1}{b-a}[f(x_1) + f(x_2) + \cdots + f(x_n)]\left(\frac{b-a}{n}\right) && \text{factor out the expression } \frac{b-a}{n} \\
&= \frac{1}{b-a}[f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x && \text{since } \Delta x = \frac{b-a}{n} \\
&= \frac{1}{b-a}\sum_{j=1}^{n} f(x_j)\Delta x
\end{aligned}$$

which we recognize as a Riemann sum.

If we refine the partition of the interval $a \le x \le b$ by taking more and more subdivision points, then $V_n$ becomes more and more like what we may intuitively think of as the average value $V$ of $f(x)$ over the entire interval $a \le x \le b$. Thus, it is reasonable to *define* the average value $V$ as the limit of the Riemann sum $V_n$ as $n \to +\infty$; that is, as the definite integral

$$\begin{aligned}
V &= \lim_{n\to+\infty} V_n = \lim_{n\to+\infty} \frac{1}{b-a}\sum_{j=1}^{n} f(x_j)\,\Delta x \\
&= \frac{1}{b-a}\int_a^b f(x)\,dx
\end{aligned}$$

To summarize:

**The Average Value of a Function** ■ Let $f(x)$ be a function that is continuous on the interval $a \le x \le b$. Then the *average value* $V$ of $f(x)$ over $a \le x \le b$ is given by the definite integral

$$V = \frac{1}{b-a}\int_a^b f(x)\,dx$$

## EXAMPLE 5.4.5

A manufacturer determines that $t$ months after introducing a new product, the company's sales will be $S(t)$ thousand dollars, where

$$S(t) = \frac{750t}{\sqrt{4t^2 + 25}}$$

What are the average monthly sales of the company over the first 6 months after the introduction of the new product?

### Solution

The average monthly sales $V$ over the time period $0 \le t \le 6$ is given by the integral

$$V = \frac{1}{6-0}\int_0^6 \frac{750t}{\sqrt{4t^2 + 25}}\,dt$$

To evaluate this integral, make the substitution

$$\begin{aligned} u &= 4t^2 + 25 && \text{limits of integration:} \\ du &= 4(2t\,dt) && \text{if } t = 0, \text{ then } u = 4(0)^2 + 25 = 25 \\ t\,dt &= \frac{1}{8}\,du && \text{if } t = 6, \text{ then } u = 4(6)^2 + 25 = 169 \end{aligned}$$

You obtain

$$\begin{aligned} V &= \frac{1}{6}\int_0^6 \frac{750}{\sqrt{4t^2+25}}\,(t\,dt) \\ &= \frac{1}{6}\int_{25}^{169} \frac{750}{\sqrt{u}}\left(\frac{1}{8}\,du\right) = \frac{750}{6(8)}\int_{25}^{169} u^{-1/2}\,du \\ &= \frac{750}{6(8)}\left(\frac{u^{1/2}}{1/2}\right)\Bigg|_{25}^{169} = \frac{750(2)}{6(8)}\left[(169)^{1/2} - (25)^{1/2}\right] \\ &= 250 \end{aligned}$$

Thus, for the 6-month period immediately after the introduction of the new product, the company's sales average \$250,000 per month.

## EXAMPLE 5.4.6

A researcher models the temperature $T$ (in °C) during the time period from 6 A.M. to 6 P.M. in a certain northern city by the function

$$T(t) = 3 - \frac{1}{3}(t-4)^2 \qquad \text{for } 0 \le t \le 12$$

where $t$ is the number of hours after 6 A.M.

**a.** What is the average temperature in the city during the workday, from 8 A.M. to 5 P.M.?

**b.** At what time (or times) during the workday is the temperature in the city the same as the average temperature found in part (a)?

### Solution

**a.** Since 8 A.M. and 5 P.M. are, respectively, $t = 2$ and $t = 11$ hours after 6 A.M., we want to compute the average of the temperature $T(t)$ for $2 \le t \le 11$, which is given by the definite integral

$$\begin{aligned} T_{\text{ave}} &= \frac{1}{11-2}\int_2^{11}\left[3 - \frac{1}{3}(t-4)^2\right]dt \\ &= \frac{1}{9}\left[3t - \frac{1}{3}\frac{1}{3}(t-4)^3\right]\Bigg|_2^{11} \\ &= \frac{1}{9}\left[3(11) - \frac{1}{9}(11-4)^3\right] - \frac{1}{9}\left[3(2) - \frac{1}{9}(2-4)^3\right] \\ &= -\frac{4}{3} \approx -1.33 \end{aligned}$$

Thus, the average temperature during the workday is approximately −1.33°C (or 29.6°F).

**Just-In-Time REVIEW**

Fahrenheit temperature $F$ is related to Celsius temperature $C$ by the formula

$F = \frac{9}{5}C + 32$

**b.** We want to find a time $t = t_a$ with $2 \le t_a \le 11$ such that $T(t_a) = -\frac{4}{3}$. Solving this equation, we find that

$$3 - \frac{1}{3}(t_a - 4)^2 = -\frac{4}{3}$$

$$-\frac{1}{3}(t_a - 4)^2 = -\frac{4}{3} - 3 = -\frac{13}{3} \qquad \text{subtract 3 from both sides}$$

$$(t_a - 4)^2 = (-3)\left(-\frac{13}{3}\right) = 13 \qquad \text{multiply both sides by } -3$$

$$t_a - 4 = \pm\sqrt{13} \qquad \text{take square roots on both sides}$$

$$t_a = 4 \pm \sqrt{13}$$

$$\approx 0.39 \quad \text{or} \quad 7.61$$

**Just-In-Time REVIEW**

Since there are 60 minutes in an hour, 0.61 hour is the same as 0.61(60) $\approx$ **37** minutes. Thus, 7.61 hours after 6 A.M. is 37 minutes past 1 P.M. or 1.37 P.M.

Since $t = 0.39$ is outside the time interval $2 \le t_a \le 11$ (8 A.M. to 5 P.M.), it follows that the temperature in the city is the same as the average temperature only when $t = 7.61$, that is, at approximately 1:37 P.M.

## Two Interpretations of Average Value

The average value of a function has several useful interpretations. First, note that if $f(x)$ is continuous on the interval $a \le x \le b$ and $F(x)$ is any antiderivative of $f(x)$ over the same interval, then the average value $V$ of $f(x)$ over the interval satisfies

$$V = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{b-a}[F(b) - F(a)] \qquad \text{fundamental theorem of calculus}$$

$$= \frac{F(b) - F(a)}{b-a}$$

We recognize this difference quotient as the average rate of change of $F(x)$ over $a \le x \le b$ (see Section 2.1). Thus, we have this interpretation:

**Rate Interpretation of Average Value** ■ The average value of a function $f(x)$ over an interval $a \le x \le b$ where $f(x)$ is continuous is the same as the average rate of change of any antiderivative $F(x)$ of $f(x)$ over the same interval.

For instance, since the total cost $C(x)$ of producing $x$ units of a commodity is an antiderivative of marginal cost $C'(x)$, it follows that the *average rate of change of cost over a range of production $a \le x \le b$ equals the average value of the marginal cost over the same range.*

The average value of a function $f(x)$ on an interval $a \le x \le b$ where $f(x) \ge 0$ can also be interpreted geometrically by rewriting the integral formula for average value

$$V = \frac{1}{b-a}\int_a^b f(x)\,dx$$

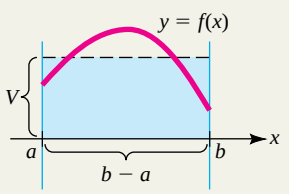
in the form

$$(b - a)V = \int_a^b f(x)\,dx$$

In the case where $f(x) \geq 0$ on the interval $a \leq x \leq b$, the integral on the right can be interpreted as the area under the curve $y = f(x)$ over $a \leq x \leq b$, and the product on the left as the area of a rectangle of height $V$ and width $b - a$ equal to the length of the interval. In other words:

**Geometric Interpretation of Average Value** ■ The average value $V$ of $f(x)$ over an interval $a \leq x \leq b$ where $f(x)$ is continuous and satisfies $f(x) \geq 0$ is equal to the height of a rectangle whose base is the interval and whose area is the same as the area under the curve $y = f(x)$ over $a \leq x \leq b$.

This geometric interpretation is illustrated in Figure 5.16.



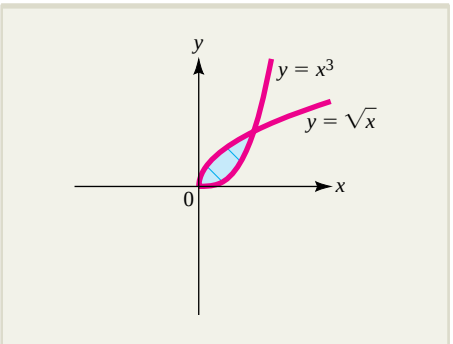

The rectangle with base $a \leq x \leq b$ and height $V$ has the same area as the region under the curve $y = f(x)$.

**FIGURE 5.16** Geometric interpretation of average value $V$.

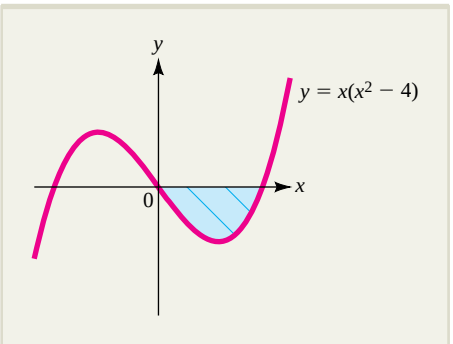
## EXERCISES ■ 5.4

*In Exercises 1 through 4, find the area of the shaded region.*

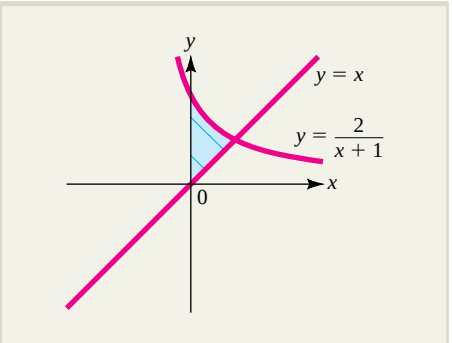
**1.**



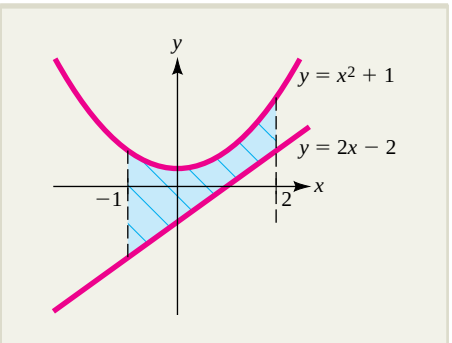

**2.**



**3.**




**4.**




*In Exercises 5 through 18, sketch the given region R and then find its area.*

**5.** $R$ is the region bounded by the lines $y = x$, $y = -x$, and $x = 1$.

**6.** $R$ is the region bounded by the curves $y = x^2$, $y = -x^2$, and the line $x = 1$.

**7.** $R$ is the region bounded by the $x$ axis and the curve $y = -x^2 + 4x - 3$.

**8.** $R$ is the region bounded by the curves $y = e^x$, $y = e^{-x}$, and the line $x = \ln 2$.

**9.** $R$ is the region bounded by the curve $y = x^2 - 2x$ and the $x$ axis. [*Hint:* Note that the region is below the $x$ axis.]

**10.** $R$ is the region bounded by the curve $y = \dfrac{1}{x^2}$ and the lines $y = x$ and $y = \dfrac{x}{8}$.

**11.** $R$ is the region bounded by the curves $y = x^2 - 2x$ and $y = -x^2 + 4$.

**12.** $R$ is the region between the curve $y = x^3$ and the line $y = 9x$, for $x \geq 0$.

**13.** $R$ is the region between the curves $y = x^3 - 3x^2$ and $y = x^2 + 5x$.

**14.** $R$ is the triangle bounded by the line $y = 4 - 3x$ and the coordinate axes.

**15.** $R$ is the triangle with vertices $(-4, 0)$, $(2, 0)$, and $(2, 6)$.

**16.** $R$ is the rectangle with vertices $(1, 0)$, $(-2, 0)$, $(-2, 5)$, and $(1, 5)$.

**17.** $R$ is the trapezoid bounded by the lines $y = x + 6$ and $x = 2$ and the coordinate axes.

**18.** $R$ is the trapezoid bounded by the lines $y = x + 2$, $y = 8 - x$, $x = 2$, and the $y$ axis.

*In Exercises 19 through 24, find the average value of the given function f(x) over the specified interval $a \leq x \leq b$.*

**19.** $f(x) = 1 - x^2$ over $-3 \leq x \leq 3$

**20.** $f(x) = x^2 - 3x + 5$ over $-1 \leq x \leq 2$

**21.** $f(x) = e^{-x}(4 - e^{2x})$ over $-1 \leq x \leq 1$

**22.** $f(x) = e^{2x} + e^{-x}$ over $0 \leq x \leq \ln 2$

**23.** $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ over $0 \leq x \leq \ln 3$

**24.** $f(x) = \dfrac{x + 1}{x^2 + 2x + 6}$ over $-1 \leq x \leq 1$

*In Exercises 25 through 28, find the average value V of the given function over the specified interval. In each case, sketch the graph of the function along with the rectangle whose base is the given interval and whose height is the average value V.*

**25.** $f(x) = 2x - x^2$ over $0 \leq x \leq 2$

**26.** $f(x) = x$ over $0 \leq x \leq 4$

**27.** $h(u) = \dfrac{1}{u}$ over $2 \leq u \leq 4$

**28.** $g(t) = e^{-2t}$ over $-1 \leq t \leq 2$

**LORENTZ CURVES** *In Exercises 29 through 34, find the Gini index for the given Lorentz curve.*

**29.** $L(x) = x^3$

**30.** $L(x) = x^2$

**31.** $L(x) = 0.55x^2 + 0.45x$

**32.** $L(x) = 0.7x^2 + 0.3x$

**33.** $L(x) = \dfrac{2}{3}x^{3.7} + \dfrac{1}{3}x$

**34.** $L(x) = \dfrac{e^x - 1}{e - 1}$

**35.** **AVERAGE SUPPLY** A manufacturer supplies $S(p) = 0.5p^2 + 3p + 7$ hundred units of a certain commodity to the market when the price is $p$ dollars per unit. Find the average supply as the price varies from $p = \$2$ to $p = \$5$.

**36.** **EFFICIENCY** After $t$ months on the job, a postal clerk can sort $Q(t) = 700 - 400e^{-0.5t}$ letters per hour. What is the average rate at which the clerk sorts mail during the first 3 months on the job?

**37.** **INVENTORY** An inventory of 60,000 kilograms of a certain commodity is used at a constant rate and is exhausted after 1 year. What is the average inventory for the year?

**38.** **FOOD PRICES** Records indicate that $t$ months after the beginning of the year, the price of ground beef in local supermarkets was

$$P(t) = 0.09t^2 - 0.2t + 4$$

dollars per pound. What was the average price of ground beef during the first 3 months of the year?

**39.** **BACTERIAL GROWTH** The number of bacteria present in a certain culture after $t$ minutes of an experiment was $Q(t) = 2{,}000e^{0.05t}$. What was the average number of bacteria present during the first 5 minutes of the experiment?

**40.** **TEMPERATURE** Records indicate that $t$ hours past midnight, the temperature at the local airport was $f(t) = -0.3t^2 + 4t + 10$ degrees Celsius. What was the average temperature at the airport between 9:00 A.M. and noon?

**41.** **INVESTMENT** Marya invests \$10,000 for 5 years in a bank that pays 5% annual interest.

**a.** What is the average value of her account over this time period if interest is compounded continuously?

**b.** How would you find the average value of the account if interest is compounded quarterly? Write a paragraph to explain how you would proceed.

**42.** **INVESTMENT** Suppose that $t$ years from now, one investment plan will be generating profit at the rate of $P_1'(t) = 100 + t^2$ hundred dollars per year, while a second investment will be generating profit at the rate of $P_2'(t) = 220 + 2t$ hundred dollars per year.

**a.** For how many years does the rate of profitability of the second investment exceed that of the first?

**b.** Compute the net excess profit assuming that you invest in the second plan for the time period determined in part (a).

**c.** Sketch the rate of profitability curves $y = P_1'(t)$ and $y = P_2'(t)$ and shade the region whose area represents the net excess profit computed in part (b).

**43.** **INVESTMENT** Answer the questions in Exercise 42 for two investments with respective rates of profitability $P_1'(t) = 130 + t^2$ hundred dollars per year and $P_2'(t) = 306 + 5t$ hundred dollars per year.

**44.** **INVESTMENT** Answer the questions in Exercise 42 for two investments with respective rates for profitability $P_1'(t) = 60e^{0.12t}$ thousand dollars per year and $P_2'(t) = 160e^{0.08t}$ thousand dollars per year.

**45.** **INVESTMENT** Answer the questions in Exercise 42 for two investments with respective rates of profitability $P_1'(t) = 90e^{0.1t}$ thousand dollars per year and $P_2'(t) = 140e^{0.07t}$ thousand dollars per year.

**46.** **EFFICIENCY** After $t$ hours on the job, one factory worker is producing $Q_1'(t) = 60 - 2(t-1)^2$ units per hour, while a second worker is producing $Q_2'(t) = 50 - 5t$ units per hour.

**a.** If both arrive on the job at 8:00 A.M., how many more units will the first worker have produced by noon than the second worker?

**b.** Interpret the answer in part (a) as the area between two curves.

**47.** **AVERAGE POPULATION** The population of a certain community $t$ years after the year 2000 is given by

$$P(t) = \frac{e^{0.2t}}{4 + e^{0.2t}} \quad \text{million people}$$

What was the average population of the community during the decade from 2000 to 2010?

**48.** **AVERAGE COST** The cost of producing $x$ units of a new product is $C(x) = 3x\sqrt{x} + 10$ hundred dollars. What is the average cost of producing the first 81 units?

**49.** **AVERAGE DRUG CONCENTRATION** A patient is injected with a drug, and $t$ hours later, the concentration of the drug remaining in the patient's bloodstream is given by

$$C(t) = \frac{3t}{(t^2 + 36)^{3/2}} \quad \text{mg/cm}^3$$

What is the average concentration of drug during the first 8 hours after the injection?

**50. AVERAGE PRODUCTION** A company determines that if $L$ worker-hours of labor are employed, then $Q$ units of a particular commodity will be produced, where

$$Q(L) = 500L^{2/3}$$

**a.** What is the average production as labor varies from 1,000 to 2,000 worker-hours?

**b.** What labor level between 1,000 and 2,000 worker-hours results in the average production found in part (a)?

**51. TEMPERATURE** A researcher models the temperature $T$ (in °C) during the time period from 6 A.M. to 6 P.M. in a certain northern city by the function

$$T(t) = 3 - \frac{1}{3}(t-5)^2 \quad \text{for } 0 \le t \le 12$$

where $t$ is the number of hours after 6 A.M.

**a.** What is the average temperature in the city during the workday, from 8 A.M. to 5 P.M.?

**b.** At what time (or times) during the workday is the temperature in the city the same as the average temperature found in part (a)?

**52. ADVERTISING** An advertising firm is hired to promote a new television show for 3 weeks before its debut and 2 weeks afterward. After $t$ weeks of the advertising campaign, it is found that $P(t)$ percent of the viewing public is aware of the show, where

$$P(t) = \frac{59t}{0.7t^2 + 16} + 6$$

**a.** What is the average percentage of the viewing public that is aware of the show during the 5 weeks of the advertising campaign?

**b.** At what time during the first 5 weeks of the campaign is the percentage of viewers the same as the average percentage found in part (a)?

**53. TRAFFIC MANAGEMENT** For several weeks, the highway department has been recording the speed of freeway traffic flowing past a certain downtown exit. The data suggest that between the hours of 1:00 and 6:00 P.M. on a normal weekday, the speed of traffic at the exit is approximately $S(t) = t^3 - 10.5t^2 + 30t + 20$ miles per hour, where $t$ is the number of hours past noon.

**a.** Compute the average speed of the traffic between the hours of 1:00 and 6:00 P.M.

**b.** At what time between 1:00 and 6:00 P.M. is the traffic speed at the exit the same as the average speed found in part (a)?

**54. AVERAGE AEROBIC RATING** The aerobic rating of a person $x$ years old is given by

$$A(x) = \frac{110(\ln x - 2)}{x} \qquad \text{for } x \ge 10$$

What is a person's average aerobic rating from age 15 to age 25? From age 60 to age 70?

**55. THERMAL EFFECT OF FOOD** Normally, the metabolism of an organism functions at an essentially constant rate, called the *basal metabolic rate* of the organism. However, the metabolic rate may increase or decrease depending on the activity of the organism. In particular, after ingesting nutrients, the organism often experiences a surge in its metabolic rate, which then gradually returns to the basal level.

Michelle has just finished her Thanksgiving dinner, and her metabolic rate has surged from its basal level $M_0$. She then "works off" the meal over the next 12 hours. Suppose that $t$ hours after the meal, her metabolic rate is given by

$$M(t) = M_0 + 50te^{-0.1t^2} \qquad 0 \le t \le 12$$

kilojoules per hour (kJ/hr).

**a.** What is Michelle's average metabolic rate over the 12-hour period?

**b.** Sketch the graph of $M(t)$. What is the peak metabolic rate and when does it occur? [*Note:* Both the graph and the peak rate will involve $M_0$.]

**56. DISTRIBUTION OF INCOME** In a certain state, it is found that the distribution of income for lawyers is given by the Lorentz curve $y = L_1(x)$, where

$$L_1(x) = \frac{4}{5}x^2 + \frac{1}{5}x$$

while that of surgeons is given by $y = L_2(x)$, where

$$L_2(x) = \frac{5}{8}x^4 + \frac{3}{8}x$$

Compute the Gini index for each Lorentz curve. Which profession has the more equitable income distribution?

**57. DISTRIBUTION OF INCOME** Suppose a study indicates that the distribution of income for professional baseball players is given by the Lorentz curve $y = L_1(x)$, where

$$L_1(x) = \frac{2}{3}x^3 + \frac{1}{3}x$$

while those of professional football players and basketball players are $y = L_2(x)$ and $y = L_3(x)$, respectively, where

$$L_2(x) = \frac{5}{6}x^2 + \frac{1}{6}x$$

and

$$L_3(x) = \frac{3}{5}x^4 + \frac{2}{5}x$$

Find the Gini index for each professional sport and determine which has the most equitable income distribution. Which has the least equitable distribution?

**58.** **COMPARATIVE GROWTH** The population of a third-world country grows exponentially at the unsustainable rate of

$$P_1'(t) = 10e^{0.02t} \text{ thousand people per year}$$

where $t$ is the number of years after 2000. A study indicates that if certain socioeconomic changes are instituted in this country, then the population will instead grow at the restricted rate

$$P_2'(t) = 10 + 0.02t + 0.002t^2$$

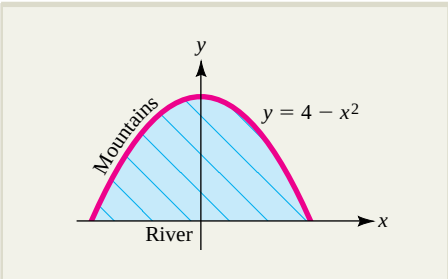
thousand people per year. How much smaller will the population of this country be in the year 2010 if the changes are made than if they are not?

**59.** **COMPARATIVE GROWTH** A second study of the country in Exercise 58 indicates that natural restrictive forces are at work that make the actual rate of growth

$$P_3'(t) = \frac{20e^{0.02t}}{1 + e^{0.02t}}$$

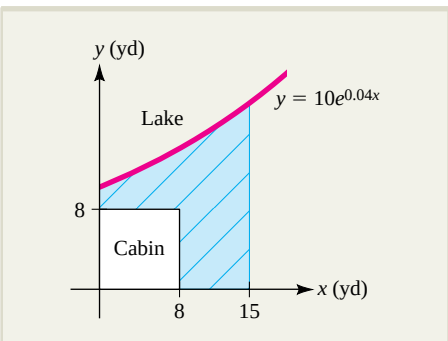
instead of the exponential rate $P_1'(t) = 10e^{0.02t}$. If this rate is correct, how much smaller will the population be in the year 2010 than if the exponential rate were correct?

**60.** **REAL ESTATE** The territory occupied by a certain community is bounded on one side by a river and on all other sides by mountains, forming the shaded region shown in the accompanying figure. If a coordinate system is set up as indicated, the mountainous boundary is given roughly by the curve $y = 4 - x^2$, where $x$ and $y$ are measured in miles. What is the total area occupied by the community?




EXERCISE 60

**61.** **REAL ESTATE EVALUATION** A square cabin with a plot of land adjacent to a lake is shown in the accompanying figure. If a coordinate system is set up as indicated, with distances given in yards, the lakefront boundary of the property is part of the curve $y = 10e^{0.04x}$. Assuming that the cabin costs \$2,000 per square yard and the land in the plot outside the cabin (the shaded region in the figure) costs \$800 per square yard, what is the total cost of this vacation property?




EXERCISE 61

**62.** **VOLUME OF BLOOD DURING SYSTOLE** A model* of the cardiovascular system relates the stroke volume $V(t)$ of blood in the aorta at time $t$ during systole (the contraction phase) to the pressure $P(t)$ in the aorta at the same time by the equation

$$V(t) = [C_1 + C_2P(t)]\left(\frac{3t^2}{T^2} - \frac{2t^3}{T^3}\right)$$

*J. G. Defares, J. J. Osborn, and H. H. Hura, *Acta Physiol. Pharm. Neerl.*, Vol. 12, 1963, pp. 189–265.

where $C_1$ and $C_2$ are positive constants and $T$ is the period of the systolic phase (a fixed time). Assume that aortic pressure $P(t)$ rises at a constant rate from $P_0$ when $t = 0$ to $P_1$ when $t = T$.

**a.** Show that

$$P(t) = \left(\frac{P_1 - P_0}{T}\right)t + P_0$$

**b.** Find the average volume of blood in the aorta during the systolic phase ($0 \le t \le T$). [*Note:* Your answer will be in terms of $C_1$, $C_2$, $P_0$, $P_1$, and $T$.]

**63.** **REACTION TO MEDICATION** In certain biological models, the human body's reaction to a particular kind of medication is measured by a function of the form

$$F(M) = \frac{1}{3}(kM^2 - M^3) \qquad 0 \le M \le k$$

where $k$ is a positive constant and $M$ is the amount of medication absorbed in the blood. The sensitivity of the body to the medication is measured by the derivative $S = F'(M)$.

**a.** Show that the body is most sensitive to the medication when $M = \frac{k}{3}$.

**b.** What is the average reaction to the medication for $0 \le M \le \frac{k}{3}$?

**64.** Use the graphing utility of your calculator to draw the graphs of the curves $y = x^2e^{-x}$ and $y = \frac{1}{x}$ on the same screen. Use **ZOOM** and **TRACE** or some other feature of your calculator to find where the curves intersect. Then compute the area of the region bounded by the curves using the numeric integration feature.

**65.** Repeat Exercise 64 for the curves

$$\frac{x^2}{5} - \frac{y^2}{2} = 1 \quad \text{and} \quad y = x^3 - 8.9x^2 + 26.7x - 27$$

**66.** Show that the average value $V$ of a continuous function $f(x)$ over the interval $a \le x \le b$ may be computed as the slope of the line joining the points $(a, F(a))$ and $(b, F(b))$ on the curve $y = F(x)$, where $F(x)$ is any antiderivative of $f(x)$ over $a \le x \le b$.

**67.** Consider an object moving along a straight line. Explain why the object's average velocity over any time interval equals the average value of its velocity over that interval.

## SECTION 5.5 Additional Applications to Business and Economics

In this section, we examine several important applications of definite integration to business and economics, such as future and present value of an income flow, consumers' willingness to spend, and consumers' and producers' surplus. We begin by showing how integration can be used to measure the value of an asset.

### Useful Life of a Machine

Suppose that $t$ years after being put into use, a machine has generated total revenue $R(t)$ and that the total cost of operating and servicing the machine up to this time is $C(t)$. Then the total profit generated by the machine up to time $t$ is $P(t) = R(t) - C(t)$. Profit declines when costs accumulate at a higher rate than revenue; that is, when $C'(t) > R'(t)$. Thus, a manager may consider disposing of the machine at the time $T$ when $C'(T) = R'(T)$, and for this reason, the time period $0 \le t \le T$ is called the **useful life** of the machine. The net profit over the useful life of the machine provides the manager with a measure of its value.

### EXAMPLE 5.5.1

Suppose that when it is $t$ years old, a particular industrial machine is generating revenue at the rate $R'(t) = 5{,}000 - 20t^2$ dollars per year and that operating and

**EXPLORE!**

Refer to Example 5.5.1. Suppose a new cost rate function $C'_{\text{new}}(t) = 2{,}000 + 6t^2$ is in place. Compare its useful life and net profit with those of the original cost rate function. Use the window [0, 20]5 by [−2,000, 8,000]1,000.

servicing costs related to the machine are accumulating at the rate $C'(t) = 2{,}000 + 10t^2$ dollars per year.

**a.** What is the useful life of this machine?

**b.** Compute the net profit generated by the machine over its period of useful life.

## Solution

**a.** To find the machine's useful life $T$, solve

$$\begin{aligned} C'(t) &= R'(t) \\ 2{,}000 + 10t^2 &= 5{,}000 - 20t^2 \\ 30t^2 &= 3{,}000 \\ t^2 &= 100 \\ t &= 10 \end{aligned}$$

Thus, the machine has a useful life of $T = 10$ years.

**b.** Since profit $P(t)$ is given by $P(t) = R(t) - C(t)$, we have $P'(t) = R'(t) - C'(t)$, and the net profit generated by the machine over its useful life $0 \le t \le 10$ is

$$\begin{aligned} \text{NP} = P(10) - P(0) &= \int_0^{10} P'(t)\,dt \\ &= \int_0^{10} [R'(t) - C'(t)]\,dt \\ &= \int_0^{10} [(5{,}000 - 20t^2) - (2{,}000 + 10t^2)]\,dt \\ &= \int_0^{10} [3{,}000 - 30t^2]\,dt \\ &= 3{,}000t - 10t^3 \Big|_0^{10} = \$20{,}000 \end{aligned}$$

The rate of revenue and rate of cost curves are sketched in Figure 5.17. The net earnings of the machine over its useful life are represented by the area of the (shaded) region between the two rate curves.

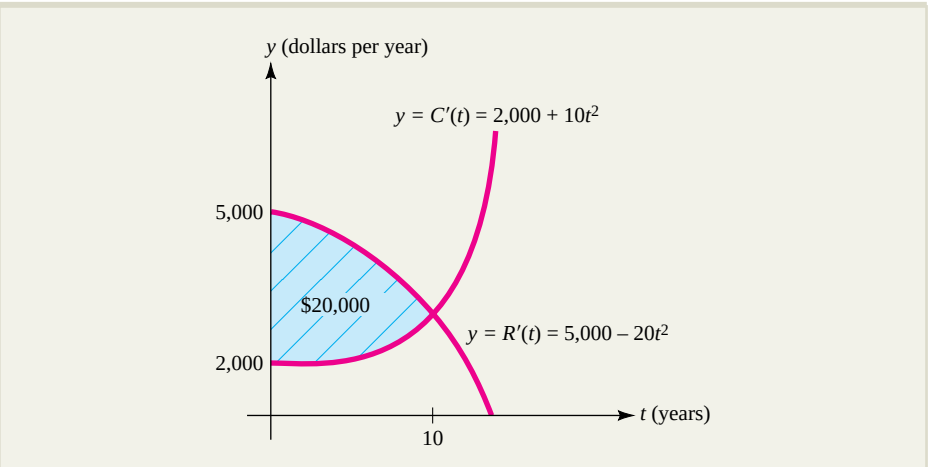



**FIGURE 5.17** Net profit over the useful life of a machine.

## Future Value and Present Value of an Income Flow

In our next application, we consider a stream of income transferred continuously into an account in which it earns interest over a specified time period (the **term**). Then the **future value of the income stream** is the total amount (money transferred into the account plus interest) that is accumulated in this way during the specified term.

The calculation of the amount of an income stream is illustrated in Example 5.5.2. The strategy is to approximate the continuous income stream by a sequence of discrete deposits called an **annuity.** The amount of the approximating annuity is a certain sum whose limit (a definite integral) is the future value of the income stream.

### EXAMPLE 5.5.2

Money is transferred continuously into an account at the constant rate of \$1,200 per year. The account earns interest at the annual rate of 8% compounded continuously. How much will be in the account at the end of 2 years?

#### Solution

Recall from Section 4.1 that $P$ dollars invested at 8% compounded continuously will be worth $Pe^{0.08t}$ dollars $t$ years later.

To approximate the future value of the income stream, divide the 2-year time interval $0 \le t \le 2$ into $n$ equal subintervals of length $\Delta t$ years and let $t_j$ denote the beginning of the $j$th subinterval. Then, during the $j$th subinterval (of length $\Delta t$ years),

$$\text{Money deposited} = (\text{dollars per year})(\text{number of years}) = 1{,}200\,\Delta t$$

If all of this money were deposited at the beginning of the subinterval (at time $t_j$), it would remain in the account for $2 - t_j$ years and therefore would grow to $(1{,}200\,\Delta t)e^{0.08(2-t_j)}$ dollars. Thus,

$$\begin{array}{c}\text{Future value of money deposited}\\ \text{during } j\text{th subinterval}\end{array} \approx 1{,}200e^{0.08(2-t_j)}\Delta t$$

The situation is illustrated in Figure 5.18.

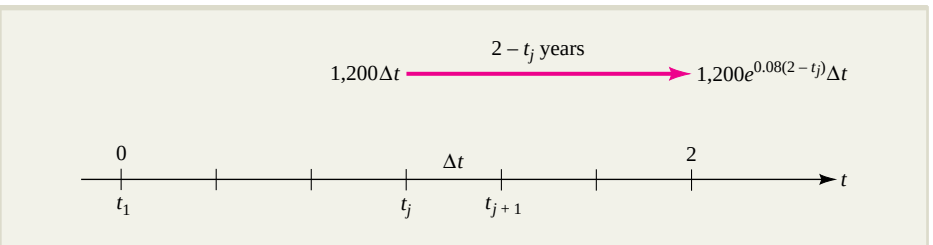



**FIGURE 5.18** The (approximate) future value of the money deposited during the $j$th subinterval.

The future value of the entire income stream is the sum of the future values of the money deposited during each of the $n$ subintervals. Hence,

$$\text{Future value of income stream} \approx \sum_{j=1}^{n} 1{,}200e^{0.08(2-t_j)}\Delta t$$

(Note that this is only an approximation because it is based on the assumption that all $1{,}200\,\Delta t_n$ dollars are deposited at time $t_j$ rather than continuously throughout the $j$th subinterval.)

As $n$ increases without bound, the length of each subinterval approaches zero and the approximation approaches the true future value of the income stream. Hence,

$$\begin{aligned}\text{Future value of income stream} &= \lim_{n\to+\infty}\sum_{j=1}^{n} 1{,}200e^{0.08(2-t_j)}\Delta t\\ &= \int_0^2 1{,}200e^{0.08(2-t)}\,dt = 1{,}200e^{0.16}\int_0^2 e^{-0.08t}\,dt\\ &= -\frac{1{,}200}{0.08}e^{0.16}(e^{-0.08t})\Big|_0^2 = -15{,}000e^{0.16}(e^{-0.16}-1)\\ &= -15{,}000 + 15{,}000e^{0.16} \approx \$2{,}602.66\end{aligned}$$

By generalizing the reasoning illustrated in Example 5.5.2, we are led to this integration formula for the future value of an income stream with rate of flow given by $f(t)$ for a term of $T$ years:

$$\begin{aligned}\text{FV} &= \int_0^T f(t)\,e^{r(T-t)}\,dt\\ &= \int_0^T f(t)\,e^{rT}\,e^{-rt}\,dt\\ &= e^{rT}\int_0^T f(t)\,e^{-rt}\,dt \quad \text{factor constant } e^{rT} \text{ outside integral}\end{aligned}$$

The first and last forms of the formula for future value are both listed next for future reference.

**Future Value of an Income Stream** ■ Suppose money is being transferred continuously into an account over a time period $0 \le t \le T$ at a rate given by the function $f(t)$ and that the account earns interest at an annual rate $r$ compounded continuously. Then the future value FV of the income stream over the term $T$ is given by the definite integral

$$\text{FV} = \int_0^T f(t)\,e^{r(T-t)}\,dt = e^{rT}\int_0^T f(t)\,e^{-rt}\,dt$$

In Example 5.5.2, we had $f(t) = 1{,}200$, $r = 0.08$, and $T = 2$, so that

$$\text{FV} = e^{0.08(2)}\int_0^2 1{,}200e^{-0.08t}\,dt$$

The **present value** of an income stream generated at a continuous rate $f(t)$ over a specified term of $T$ years is the amount of money $A$ that must be deposited now at the prevailing interest rate to generate the same income as the income

stream over the same $T$-year period. Since $A$ dollars invested at an annual interest rate $r$ compounded continuously will be worth $Ae^{rT}$ dollars in $T$ years, we must have

$$Ae^{rT} = e^{rT}\int_0^T f(t)\,e^{-rt}\,dt$$

$$A = \int_0^T f(t)\,e^{-rt}\,dt \qquad \text{divide both sides by } e^{rT}$$

To summarize:

**Present Value of an Income Stream** ■ The **present value** PV of an income stream that is deposited continuously at the rate $f(t)$ into an account that earns interest at an annual rate $r$ compounded continuously for a term of $T$ years is given by

$$\text{PV} = \int_0^T f(t)\,e^{-rt}\,dt$$

Example 5.5.3 illustrates how present value can be used in making certain financial decisions.

## EXAMPLE 5.5.3

Jane is trying to decide between two investments. The first costs \$1,000 and is expected to generate a continuous income stream at the rate of $f_1(t) = 3{,}000e^{0.03t}$ dollars per year. The second investment costs \$4,000 and is estimated to generate income at the constant rate of $f_2(t) = 4{,}000$ dollars per year. If the prevailing annual interest rate remains fixed at 5% compounded continuously over the next 5 years, which investment is better over this time period?

### Solution

The net value of each investment over the 5-year time period is the present value of the investment less its initial cost. For each investment, we have $r = 0.05$ and $T = 5$.

For the first investment:

$$\begin{aligned}
\text{PV} - \text{cost} &= \int_0^5 (3{,}000e^{0.03t})e^{-0.05t}\,dt - 1{,}000 \\
&= 3{,}000\int_0^5 e^{0.03t-0.05t}\,dt - 1{,}000 \\
&= 3{,}000\int_0^5 e^{-0.02t}\,dt - 1{,}000 \\
&= 3{,}000\left(\frac{e^{-0.02t}}{-0.02}\right)\Bigg|_0^5 - 1{,}000 \\
&= -150{,}000[e^{-0.02(5)} - e^0] - 1{,}000 \\
&= 13{,}274.39
\end{aligned}$$

For the second investment:

$$\begin{aligned}\text{PV} - \text{cost} &= \int_0^5 (4{,}000)e^{-0.05t}\,dt - 4{,}000\\ &= 4{,}000\left(\frac{e^{-0.05t}}{-0.05}\right)\Bigg|_0^5 - 4{,}000\\ &= -80{,}000[e^{-0.05(5)} - e^0] - 4{,}000\\ &= 13{,}695.94\end{aligned}$$

Thus, the net income generated by the first investment is \$13,274.39, while the second generates net income of \$13,695.94. The second investment is slightly better.

## Consumer Willingness to Spend

Recall that the consumer demand function $p = D(q)$ gives the price $p$ that must be charged for each unit of the commodity if $q$ units are to be sold (demanded). If $A(q)$ is the total amount that consumers are willing to pay for $q$ units, then the demand function can also be thought of as the rate of change of $A(q)$ with respect to $q$; that is, $A'(q) = D(q)$. Integrating and assuming that $A(0) = 0$ (consumers are willing to pay nothing for 0 units), we find that $A(q_0)$, the amount that consumers are willing to pay for $q_0$ units of the commodity, is given by

$$A(q_0) = A(q_0) - A(0) = \int_0^{q_0} \frac{dA}{dq}\,dq = \int_0^{q_0} D(q)\,dq$$

In this context, economists call $A(q)$ the **total willingness to spend** and $D(q) = A'(q)$, the **marginal willingness to spend.** In geometric terms, the total willingness to spend for $q_0$ units is the area under the demand curve $p = D(q)$ between $q = 0$ and $q = q_0$, as shown in Figure 5.19.

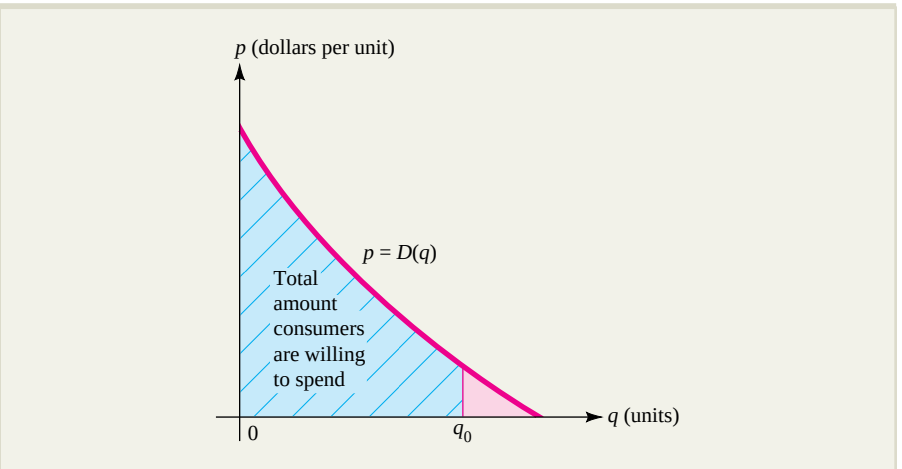



**FIGURE 5.19** The amount consumers are willing to spend is the area under the demand curve.

**EXPLORE!**

In Example 5.5.4, change $D(q)$ to $D_{new}(q) = 4(23 - q^2)$. Will the amount of money consumers are willing to spend to obtain 3 units of the commodity increase or decrease? Graph $D_{new}(q)$ in bold to compare with the graph of $D(q)$, using the viewing window [0, 5]1 by [0, 150]10.

## EXAMPLE 5.5.4

Suppose that the consumers' demand function for a certain commodity is $D(q) = 4(25 - q^2)$ dollars per unit.

**a.** Find the total amount of money consumers are willing to spend to get 3 units of the commodity.

**b.** Sketch the demand curve and interpret the answer to part (a) as an area.

### Solution

**a.** Since the demand function is $D(q) = 4(25 - q^2)$, is the total amount that consumers are willing to spend to get 3 units of the commodity is given by the definite integral

$$A(3) = \int_0^3 D(q)\,dq = 4\int_0^3 (25 - q^2)\,dq$$
$$= 4\left(25q - \frac{1}{3}q^3\right)\Big|_0^3 = \$264$$

**b.** The consumers' demand curve is sketched in Figure 5.20. In geometric terms, the total amount, \$264, that consumers are willing to spend to get 3 units of the commodity is the area under the demand curve from $q = 0$ to $q = 3$.

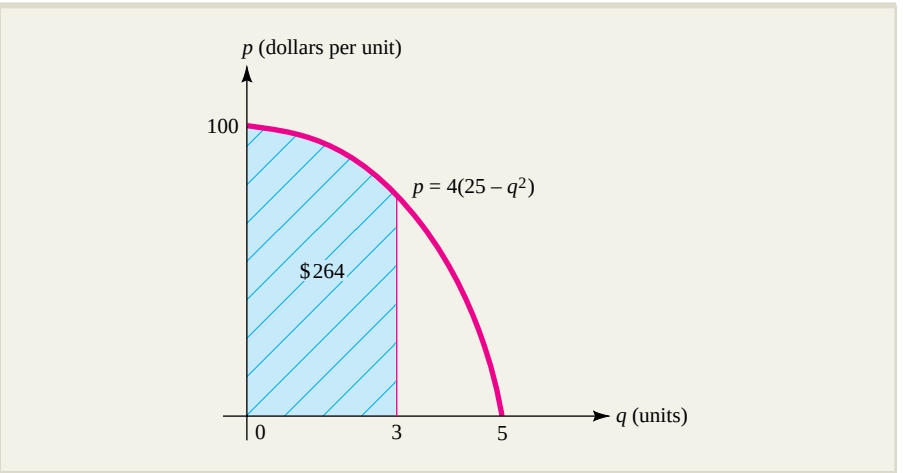



**FIGURE 5.20** Consumers' willingness to spend for 3 units when demand is given by $D(q) = 4(25 - q^2)$.

## Consumers' and Producers' Surplus

In a competitive economy, the total amount that consumers actually spend on a commodity is usually less than the total amount they would have been willing to spend. Suppose the market price of a particular commodity has been fixed at $p_0$ and consumers will buy $q_0$ units at that price. Market conditions determine that $p_0 = D(q_0)$, where $D(q)$ is the demand function for the commodity. Then the difference between the consumers' willingness to pay for $q_0$ units and the amount they *actually pay*, $p_0q_0$,

represents a perceived advantage to the consumer that economists call **consumers' surplus**. That is,

$$\begin{bmatrix}\text{Consumers'}\\ \text{surplus}\end{bmatrix} = \begin{bmatrix}\text{total amount consumers}\\ \text{would be willing to spend}\end{bmatrix} - \begin{bmatrix}\text{actual consumer}\\ \text{expenditure}\end{bmatrix}$$

To get a better feel for the concept of consumers' surplus, consider a family that is willing to spend up to \$150 to own one television set but is willing to pay no more than \$75 for a second set, say, to settle conflicts over viewing preferences. Suppose the market price for television sets turns out to be \$100 per set. Then the family spends $2 \times \$100 = \$200$ for its two sets, rather than the $\$150 + \$75 = \$225$ that it was willing to pay. The perceived savings of $\$225 - \$200 = \$25$ is the consumers' surplus of the family.

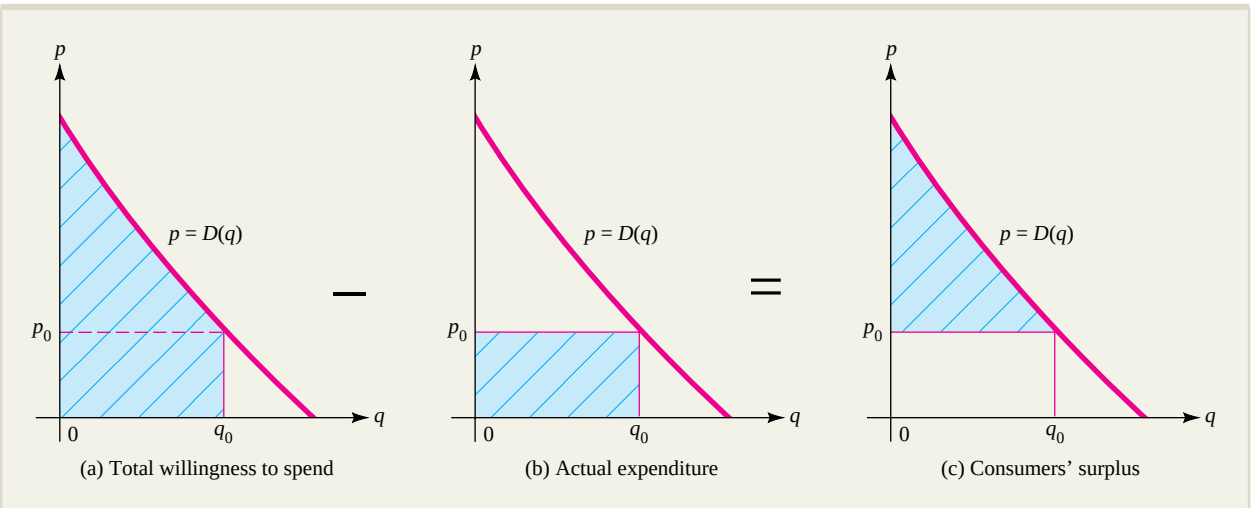
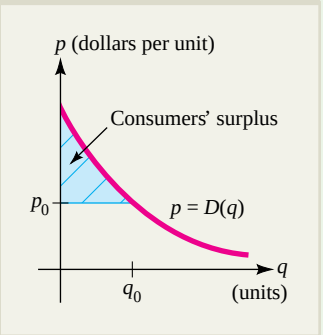



**FIGURE 5.21** Geometric interpretation of consumers' surplus.

Consumers' surplus has a simple geometric interpretation, which is illustrated in Figure 5.21. The symbols $p_0$ and $q_0$ denote the market price and corresponding demand, respectively. Figure 5.21a shows the region under the demand curve from $q = 0$ to $q = q_0$. Its area, as we have seen, represents the total amount that consumers are willing to spend to get $q_0$ units of the commodity. The rectangle in Figure 5.21b has an area of $p_0q_0$ and hence represents the actual consumer expenditure for $q_0$ units at $p_0$ dollars per unit. The difference between these two areas (Figure 5.21c) represents the consumers' surplus. That is, consumers' surplus CS is the area of the region between the demand curve $p = D(q)$ and the horizontal line $p = p_0$ and hence is equal to the definite integral

$$\begin{aligned}\text{CS} = \int_0^{q_0} [D(q) - p_0]\,dq &= \int_0^{q_0} D(q)\,dq - \int_0^{q_0} p_0\,dq\\ &= \int_0^{q_0} D(q)\,dq - p_0q\Big|_0^{q_0}\\ &= \int_0^{q_0} D(q)\,dq - p_0q_0\end{aligned}$$

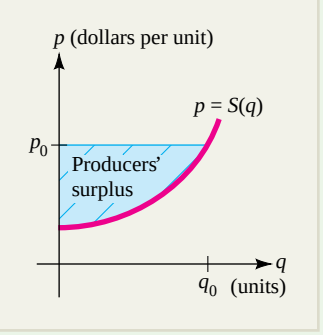



**Consumers' Surplus** ■ If $q_0$ units of a commodity are sold at a price of $p_0$ per unit and if $p = D(q)$ is the consumers' demand function for the commodity, then

$$\begin{bmatrix}\text{Consumers'}\\ \text{surplus}\end{bmatrix} = \begin{bmatrix}\text{total amount consumers}\\ \text{are willing to spend}\\ \text{for } q_0 \text{ units}\end{bmatrix} - \begin{bmatrix}\text{actual consumer}\\ \text{expenditure}\\ \text{for } q_0 \text{ units}\end{bmatrix}$$

$$\text{CS} = \int_0^{q_0} D(q)\,dq - p_0 q_0$$

**Producers' surplus** is the other side of the coin from consumers' surplus. Recall that the **supply function** $p = S(q)$ gives the price per unit that producers are willing to accept in order to supply $q$ units to the marketplace. However, any producer who is willing to accept less than $p_0 = S(q_0)$ dollars for $q_0$ units gains from the fact that the price is $p_0$. Then producers' surplus is the difference between what producers would be willing to accept for supplying $q_0$ units and the price they actually receive. Reasoning as we did with consumers' surplus, we obtain the following formula for producers' surplus.




**Producers' Surplus** ■ If $q_0$ units of a commodity are sold at a price of $p_0$ dollars per unit and $p = S(q)$ is the producers' supply function for the commodity, then

$$\begin{bmatrix}\text{Producers'}\\ \text{surplus}\end{bmatrix} = \begin{bmatrix}\text{actual consumer}\\ \text{expenditure}\\ \text{for } q_0 \text{ units}\end{bmatrix} - \begin{bmatrix}\text{total amount producers}\\ \text{receive when } q_0\\ \text{units are supplied}\end{bmatrix}$$

$$\text{PS} = p_0 q_0 - \int_0^{q_0} S(q)\,dq$$

## EXAMPLE 5.5.5

A tire manufacturer estimates that $q$ (thousand) radial tires will be purchased (demanded) by wholesalers when the price is

$$p = D(q) = -0.1q^2 + 90$$

dollars per tire, and the same number of tires will be supplied when the price is

$$p = S(q) = 0.2q^2 + q + 50$$

dollars per tire.

**a.** Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price.

**b.** Determine the consumers' and producers' surplus at the equilibrium price.

## Solution

**a.** The supply and demand curves are shown in Figure 5.22. Supply equals demand when

$$\begin{aligned} -0.1q^2 + 90 &= 0.2q^2 + q + 50 \\ 0.3q^2 + q - 40 &= 0 \\ q &= 10 \quad (\text{reject } q \approx -13.33) \end{aligned}$$

and $p = -0.1(10)^2 + 90 = 80$ dollars per tire. Thus, equilibrium occurs at a price of \$80 per tire, and then 10,000 tires are supplied and demanded.

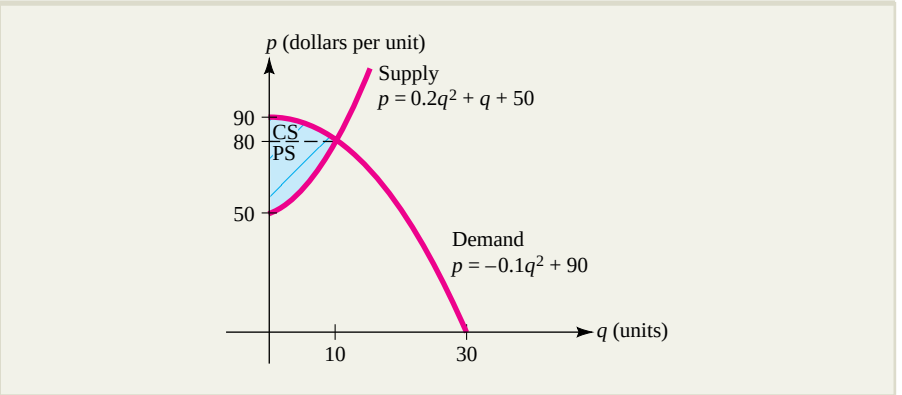



**FIGURE 5.22** Consumers' surplus and producers' surplus for the demand and supply functions in Example 5.5.5.

**b.** Using $p_0 = 80$ and $q_0 = 10$, we find that the consumers' surplus is

$$\begin{aligned} \text{CS} &= \int_0^{10} (-0.1q^2 + 90)\, dq - (80)(10) \\ &= \left[ -0.1\left(\frac{q^3}{3}\right) + 90q \right]\Bigg|_0^{10} - (80)(10) \\ &\approx 866.67 - 800 = 66.67 \end{aligned}$$

or \$66,670 (since $q_0 = 10$ is really 10,000). The consumers' surplus is the area of the shaded region labeled CS in Figure 5.22.

The producers' surplus is

$$\begin{aligned} \text{PS} &= (80)(10) - \int_0^{10} (0.2q^2 + q + 50)\, dq \\ &= (80)(10) - \left[ 0.2\left(\frac{q^3}{3}\right) + \left(\frac{q^2}{2}\right) + 50q \right]\Bigg|_0^{10} \\ &\approx 800 - 616.67 = 183.33 \end{aligned}$$

or \$183,330. The producers' surplus is the area of the shaded region labeled PS in Figure 5.22.

# EXERCISES ■ 5.5

**CONSUMERS' WILLINGNESS TO SPEND** *For the consumers' demand functions $D(q)$ in Exercises 1 through 6:*

*(a) Find the total amount of money consumers are willing to spend to get $q_0$ units of the commodity.*

*(b) Sketch the demand curve and interpret the consumers' willingness to spend in part (a) as an area.*

1. $D(q) = 2(64 - q^2)$ dollars per unit; $q_0 = 6$ units
2. $D(q) = \dfrac{300}{(0.1q + 1)^2}$ dollars per unit; $q_0 = 5$ units
3. $D(q) = \dfrac{400}{0.5q + 2}$ dollars per unit; $q_0 = 12$ units
4. $D(q) = \dfrac{300}{4q + 3}$ dollars per unit; $q_0 = 10$ units
5. $D(q) = 40e^{-0.05q}$ dollars per unit; $q_0 = 10$ units
6. $D(q) = 50e^{-0.04q}$ dollars per unit; $q_0 = 15$ units

**CONSUMERS' SURPLUS** *In Exercises 7 through 10, $p = D(q)$ is the price (dollars per unit) at which $q$ units of a particular commodity will be demanded by the market (that is, all $q$ units will be sold at this price), and $q_0$ is a specified level of production. In each case, find the price $p_0 = D(q_0)$ at which $q_0$ units will be demanded and compute the corresponding consumers' surplus CS. Sketch the demand curve $y = D(q)$ and shade the region whose area represents the consumers' surplus.*

7. $D(q) = 2(64 - q^2)$; $q_0 = 3$ units
8. $D(q) = 150 - 2q - 3q^2$; $q_0 = 6$ units
9. $D(q) = 40e^{-0.05q}$; $q_0 = 5$ units
10. $D(q) = 75e^{-0.04q}$; $q_0 = 3$ units

**PRODUCERS' SURPLUS** *In Exercises 11 through 14, $p = S(q)$ is the price (dollars per unit) at which $q$ units of a particular commodity will be supplied to the market by producers, and $q_0$ is a specified level of production. In each case, find the price $p_0 = S(q_0)$ at which $q_0$ units will be supplied and compute the corresponding producers' surplus PS. Sketch the supply curve $y = S(q)$ and shade the region whose area represents the producers' surplus.*

11. $S(q) = 0.3q^2 + 30$; $q_0 = 4$ units
12. $S(q) = 0.5q + 15$; $q_0 = 5$ units
13. $S(q) = 10 + 15e^{0.03q}$; $q_0 = 3$ units
14. $S(q) = 17 + 11e^{0.01q}$; $q_0 = 7$ units

**CONSUMERS' AND PRODUCERS' SURPLUS AT EQUILIBRIUM** *In Exercises 15 through 19, the demand and supply functions, $D(q)$ and $S(q)$, for a particular commodity are given. Specifically, $q$ thousand units of the commodity will be demanded (sold) at a price of $p = D(q)$ dollars per unit, while $q$ thousand units will be supplied by producers when the price is $p = S(q)$ dollars per unit. In each case:*

*(a) Find the equilibrium price $p_e$ (where supply equals demand).*

*(b) Find the consumers' surplus and the producers' surplus at equilibrium.*

15. $D(q) = 131 - \frac{1}{3}q^2$; $S(q) = 50 + \frac{2}{3}q^2$
16. $D(q) = 65 - q^2$; $S(q) = \frac{1}{3}q^2 + 2q + 5$
17. $D(q) = -0.3q^2 + 70$; $S(q) = 0.1q^2 + q + 20$
18. $D(q) = \sqrt{245 - 2q}$; $S(q) = 5 + q$
19. $D(q) = \dfrac{16}{q + 2} - 3$; $S(q) = \frac{1}{3}(q + 1)$
20. **PROFIT OVER THE USEFUL LIFE OF A MACHINE** Suppose that when it is $t$ years old, a particular industrial machine generates revenue at the rate $R'(t) = 6{,}025 - 8t^2$ dollars per year and that operating and servicing costs accumulate at the rate $C'(t) = 4{,}681 + 13t^2$ dollars per year.
    - **a.** How many years pass before the profitability of the machine begins to decline?
    - **b.** Compute the net profit generated by the machine over its useful lifetime.
    - **c.** Sketch the revenue rate curve $y = R'(t)$ and the cost rate curve $y = C'(t)$ and shade the region whose area represents the net profit computed in part (b).

**21. PROFIT OVER THE USEFUL LIFE OF A MACHINE** Answer the questions in Exercise 20 for a machine that generates revenue at the rate $R'(t) = 7{,}250 - 18t^2$ dollars per year and for which costs accumulate at the rate $C'(t) = 3{,}620 + 12t^2$ dollars per year.

**22. FUND-RAISING** It is estimated that $t$ weeks from now, contributions in response to a fund-raising campaign will be coming in at the rate of $R'(t) = 5{,}000e^{-0.2t}$ dollars per week, while campaign expenses are expected to accumulate at the constant rate of \$676 per week.

**a.** For how many weeks does the rate of revenue exceed the rate of cost?

**b.** What net earnings will be generated by the campaign during the period of time determined in part (a)?

**c.** Interpret the net earnings in part (b) as an area between two curves.

**23. FUND-RAISING** Answer the questions in Exercise 22 for a charity campaign in which contributions are made at the rate of $R'(t) = 6{,}537e^{-0.3t}$ dollars per week and expenses accumulate at the constant rate of \$593 per week.

**24. THE AMOUNT OF AN INCOME STREAM** Money is transferred continuously into an account at the constant rate of \$2,400 per year. The account earns interest at the annual rate of 6% compounded continuously. How much will be in the account at the end of 5 years?

**25. THE AMOUNT OF AN INCOME STREAM** Money is transferred continuously into an account at the constant rate of \$1,000 per year. The account earns interest at the annual rate of 10% compounded continuously. How much will be in the account at the end of 10 years?

**26. CONSTRUCTION DECISION** Magda wants to expand and renovate her import store and is presented with two plans for making the improvements. The first plan will cost her \$40,000 and the second will cost only \$25,000. However, she expects the improvements resulting from the first plan to provide income at the continuous rate of \$10,000 per year, while the income flow from the second plan provides \$8,000 per year. Which plan will result in more net income over the next 3 years if the prevailing rate of interest is 5% per year compounded continuously?

**27. RETIREMENT ANNUITY** At age 25, Tom starts making annual deposits of \$2,500 into an IRA account that pays interest at an annual rate of 5% compounded continuously. Assuming that his payments are made as a continuous income flow, how much money will be in his account if he retires at age 60? At age 65?

**28. RETIREMENT ANNUITY** When she is 30, Sue starts making annual deposits of \$2,000 into a bond fund that pays 8% annual interest compounded continuously. Assuming that her deposits are made as a continuous income flow, how much money will be in her account if she retires at age 55?

**29. THE PRESENT VALUE OF AN INVESTMENT** An investment will generate income continuously at the constant rate of \$1,200 per year for 5 years. If the prevailing annual interest rate remains fixed at 5% compounded continuously, what is the present value of the investment?

**30. THE PRESENT VALUE OF A FRANCHISE** The management of a national chain of fast-food outlets is selling a 10-year franchise in Cleveland, Ohio. Past experience in similar localities suggests that $t$ years from now the franchise will be generating profit at the rate of $f(t) = 10{,}000$ dollars per year. If the prevailing annual interest rate remains fixed at 4% compounded continuously, what is the present value of the franchise?

**31. INVESTMENT ANALYSIS** Adam is trying to choose between two investment opportunities. The first will cost \$50,000 and is expected to produce income at the continuous rate of \$15,000 per year. The second will cost \$30,000 and is expected to produce income at the rate of \$9,000 per year. If the prevailing rate of interest stays constant at 6% per year compounded continuously, which investment is better over the next 5 years?

**32. INVESTMENT ANALYSIS** Kevin spends \$4,000 for an investment that generates a continuous income stream at the rate of $f_1(t) = 3{,}000$ dollars per year. His friend, Molly, makes a separate investment that also generates income continuously, but at a rate of $f_2(t) = 2{,}000e^{0.04t}$ dollars per year. The couple discovers that their investments have exactly the same net value over a 4-year period. If the prevailing annual interest rate stays fixed at 5% compounded continuously, how much did Molly pay for her investment?

**33. CONSUMERS' SURPLUS** A manufacturer of machinery parts determines that $q$ units of a particular piece will be sold when the price is $p = 110 - q$ dollars per unit. The total cost of producing those $q$ units is $C(q)$ dollars, where

$$C(q) = q^3 - 25q^2 + 2q + 3{,}000$$

**a.** How much profit is derived from the sale of the $q$ units at $p$ dollars per unit? [*Hint:* First find the revenue $R = pq$; then profit = revenue − cost.]

**b.** For what value of $q$ is profit maximized?

**c.** Find the consumers' surplus for the level of production $q_0$ that corresponds to maximum profit.

**34. CONSUMERS' SURPLUS** Repeat Exercise 33 for $C(q) = 2q^3 - 59q^2 + 4q + 7{,}600$. and $p = 124 - 2q$.

**35. DEPLETION OF ENERGY RESOURCES** Oil is being pumped from an oil field $t$ years after its opening at the rate of $P'(t) = 1.3e^{0.04t}$ billion barrels per year. The field has a reserve of 20 billion barrels, and the price of oil holds steady at \$112 per barrel.

**a.** Find $P(t)$, the amount of oil pumped from the field at time $t$. How much oil is pumped from the field during the first 3 years of operation? The next 3 years?

**b.** For how many years $T$ does the field operate before it runs dry?

**c.** If the prevailing annual interest rate stays fixed at 5% compounded continuously, what is the present value of the continuous income stream $V = 112P'(t)$ over the period of operation of the field $0 \le t \le T$?

**d.** If the owner of the oil field decides to sell on the first day of operation, do you think the present value determined in part (c) would be a fair asking price? Explain your reasoning.

**36. DEPLETION OF ENERGY RESOURCES** Answer the questions in Exercise 35 for another oil field with a pumping rate of $P'(t) = 1.5e^{0.03t}$ and with a reserve of 16 billion barrels. You may assume that the price of oil is still \$112 per barrel and that the prevailing annual interest rate is 5%.

**37. DEPLETION OF ENERGY RESOURCES** Answer the questions in Exercise 35 for an oil field with a pumping rate of $P'(t) = 1.2e^{0.02t}$ and with a reserve of 12 billion barrels. Assume that the prevailing interest rate is 5% as before, but that the price of oil after $t$ years is given by $A(t) = 112e^{0.015t}$.

**38. LOTTERY PAYOUT** A \$2 million state lottery winner is given \$250,000 check now and a continous income flow at the rate of \$200,000 per year for 10 years. If the prevailing rate of interest is 5% per year compounded continously, is this a good deal for the winner or not? Explain.

**39. LOTTERY PAYOUT** The winner of a state lottery is offered a choice of either receiving \$10 million now as a lump sum or of receiving $A$ dollars a year for the next 6 years as a continuous income stream. If the prevailing annual interest rate is 5% compounded continuously and the two payouts are worth the same, what is $A$?

**40. SPORTS CONTRACTS** A star baseball free agent is the object of a bidding war between two rival teams. The first team offers a 3 million dollar signing bonus and a 5-year contract guaranteeing him 8 million dollars this year and an increase of 3% per year for the remainder of the contract. The second team offers \$9 million per year for 5 years with no incentives. If the prevailing annual interest rate stays fixed at 4% compounded continuously, which offer is worth more? [*Hint:* Assume that with both offers, the salary is paid as a continuous income stream.]

**41. PRESENT VALUE OF AN INVESTMENT** An investment produces a continuous income stream at the rate of $A(t)$ thousand dollars per year at time $t$, where

$$A(t) = 10e^{1-0.05t}$$

The prevailing rate of interest is 5% per year compounded continuously.

**a.** What is the future value of the investment over a term of 5 years ($0 \le t \le 5$)?

**b.** What is the present value of the investment over the time period $1 \le t \le 3$?

**42. PROFIT FROM AN INVENTION** A marketing survey indicates that $t$ months after a new type of computerized air purifier is introduced to the market, sales will be generating profit at the rate of $P'(t)$ thousand dollars per month, where

$$P'(t) = \frac{500[1.4 - \ln(0.5t + 1)]}{t + 2}$$

**a.** When is the rate of profitability positive and when is it negative? When is the rate increasing and when is it decreasing?

**b.** At what time $t = t_m$ is monthly profit maximized? Find the net change in profit over the time period $0 \le t \le t_m$.

**c.** It costs the manufacturer \$100,000 to develop the purifier product, so $P(0) = -100$. Use this information along with integration to find $P(t)$.

**d.** Sketch the graph of $P(t)$ for $t \ge 0$. A "fad" is a product that gains rapid success in the market, then just as quickly fades from popularity. Based on the graph $P(t)$, would you call the purifiers a "fad"? Explain.

**43. TOTAL REVENUE** Consider the following problem: A certain oil well that yields 300 barrels of crude oil a month will run dry in 3 years. It is estimated that $t$ months from now the price of crude oil will be $P(t) = 118 + 0.3\sqrt{t}$ dollars per barrel. If the oil is sold as soon as it is extracted from the ground, what will be the total future revenue from the well?

**a.** Solve the problem using definite integration. [*Hint:* Divide the 3-year (36-month) time interval $0 \le t \le 36$ into $n$ equal subintervals of length $\Delta t$ and let $t_j$ denote the beginning of the $j$th subinterval. Find an expression that estimates the revenue $R(t_j)$ obtained during the $j$th subinterval. Then express the total revenue as the limit of a sum.]

**b.** Read an article on the petroleum industry and write a paragraph on mathematical methods of modeling oil production.*

**44. INVENTORY STORAGE COSTS** A manufacturer receives $N$ units of a certain raw material that are initially placed in storage and then withdrawn and used at a constant rate until the supply is exhausted 1 year later. Suppose storage costs remain fixed at $p$ dollars per unit per year. Use definite integration to find an expression for the total storage cost the manufacturer will pay during the year. [*Hint:* Let $Q(t)$ denote the number of units in storage after $t$ years and find an expression for $Q(t)$. Then subdivide the interval $0 \le t \le 1$ into $n$ equal subintervals and express the total storage cost as the limit of a sum.]

**45. FUTURE VALUE OF AN INVESTMENT** A constant incomc stream of $M$ dollars per year is invested at an annual rate $r$ compounded continuously for a term of $T$ years. Show that the future value of such an investment is

$$\text{FV} = \frac{M}{r}(e^{rT} - 1)$$

**46. PRESENT VALUE OF AN INVESTMENT** A constant income stream of $M$ dollars per year is invested at an annual rate $r$ compounded continuously for a term of $T$ years. Show that the present value of such an investment is

$$\text{PV} = \frac{M}{r}(1 - e^{-rT})$$

*A good place to start is the article by J. A. Weyland and D. W. Ballew, "A Relevant Calculus Problem: Estimation of U.S. Oil Reserves," *The Mathematics Teacher,* Vol. 69, 1976, pp. 125–126.

## SECTION 5.6 Additional Applications to the Life and Social Sciences

We have already seen how definite integration can be used to compute quantities of interest in the social and life sciences, such as net change, average value, and the Gini index of a Lorentz curve. In this section, we examine several additional such applications, including survival and renewal within a group, blood flow through an artery, and cardiac output. We shall also discuss how volume can be computed using integration and used for purposes such as measuring the size of a lake or a tumor.

### Survival and Renewal

In Example 5.6.1, a **survival function** gives the fraction of individuals in a group or population that can be expected to remain in the group for any specified period of time. A **renewal function** giving the rate at which new members arrive is also known,

and the goal is to predict the size of the group at some future time. Problems of this type arise in many fields, including sociology, ecology, demography, and even finance, where the "population" is the number of dollars in an investment account and "survival and renewal" refer to features of an investment strategy.

## EXAMPLE 5.6.1

A new county mental health clinic has just opened. Statistics from similar facilities suggest that the fraction of patients who will still be receiving treatment at the clinic $t$ months after their initial visit is given by the function $f(t) = e^{-t/20}$. The clinic initially accepts 300 people for treatment and plans to accept new patients at the constant rate of $g(t) = 10$ patients per month. Approximately how many people will be receiving treatment at the clinic 15 months from now?

### Solution

Since $f(15)$ is the fraction of patients whose treatment continues at least 15 months, it follows that of the current 300 patients, only $300f(15)$ will still be receiving treatment 15 months from now.

To approximate the number of *new* patients who will be receiving treatment 15 months from now, divide the 15-month time interval $0 \le t \le 15$ into $n$ equal subintervals of length $\Delta t$ months and let $t_j$ denote the beginning of the $j$th subinterval. Since new patients are accepted at the rate of 10 per month, the number of new patients accepted during the $j$th subinterval is $10\Delta t$. Fifteen months from now, approximately $15 - t_j$ months will have elapsed since these $10\Delta t$ new patients had their initial visits, and so approximately $(10\Delta t)f(15 - t_j)$ of them will still be receiving treatment at that time (Figure 5.23). It follows that the total number of new patients still receiving treatment 15 months from now can be approximated by the sum

$$\sum_{j=1}^{n} 10f(15 - t_j)\Delta t$$

Adding this to the number of current patients who will still be receiving treatment in 15 months, you get

$$P \approx 300f(15) + \sum_{j=1}^{n} 10f(15 - t_j)\Delta t$$

where $P$ is the total number of *all* patients (current and new) who will be receiving treatment 15 months from now.

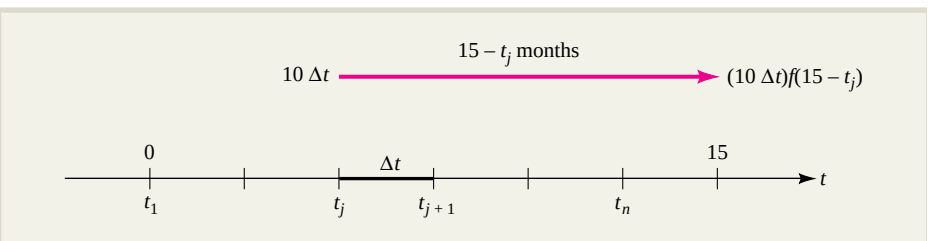



**FIGURE 5.23** New members arriving during the $j$th subinterval.

As $n$ increases without bound, the approximation improves and approaches the true value of $P$. It follows that

$$\begin{aligned} P &= 300f(15) + \lim_{n\to+\infty} \sum_{j=1}^{n} 10f(15 - t_j)\Delta t \\ &= 300f(15) + \int_0^{15} 10f(15 - t)\,dt \end{aligned}$$

Since $f(t) = e^{-t/20}$, we have $f(15) = e^{-3/4}$ and $f(15 - t) = e^{-(15-t)/20} = e^{-3/4}e^{t/20}$. Hence,

$$\begin{aligned} P &= 300e^{-3/4} + 10e^{-3/4}\int_0^{15} e^{t/20}\,dt \\ &= 300e^{-3/4} + 10e^{-3/4}\left(\frac{e^{t/20}}{1/20}\right)\Bigg|_0^{15} \\ &= 300e^{-3/4} + 200(1 - e^{-3/4}) \\ &\approx 247.24 \end{aligned}$$

That is, 15 months from now, the clinic will be treating approximately 247 patients.

In Example 5.6.1, we considered a variable survival function $f(t)$ and a constant renewal rate function $g(t)$. Essentially the same analysis applies when the renewal function also varies with time. Here is the result. Note that for definiteness, time is given in years, but the same basic formula would also apply for other units of time, for example, minutes, weeks, or months, as in Example 5.6.1.

**Survival and Renewal** ■ Suppose a population initially has $P_0$ members and that new members are added at the (renewal) rate of $R(t)$ individuals per year. Further suppose that the fraction of the population that remain for at least $t$ years after arriving is given by the (survival) function $S(t)$. Then, at the end of a term of $T$ years, the population will be

$$P(T) = P_0S(T) + \int_0^T R(t)\,S(T - t)\,dt$$

In Example 5.6.1, each time period is 1 month, the initial "population" (membership) is $P_0 = 300$, the renewal rate is $R = 10$, the survival function is $f(t) = e^{-t/20}$, and the term is $T = 15$ months. Here is another example of survival/renewal from biology.

## EXAMPLE 5.6.2

A mild toxin is introduced to a bacterial colony whose current population is 600,000. Observations indicate that $R(t) = 200e^{0.01t}$ bacteria per hour are born in the colony at time $t$ and that the fraction of the population that survives for $t$ hours after birth is $S(t) = e^{-0.015t}$. What is the population of the colony after 10 hours?

### Solution

Substituting $P_0 = 600{,}000$, $R(t) = 200e^{0.01t}$, and $S(t) = e^{-0.015t}$ into the formula for survival and renewal, we find that the population at the end of the term of $T = 10$ hours is

$$P(10) = \underbrace{600{,}000}_{P_0}\underbrace{e^{-0.015(10)}}_{S(10)} + \int_0^{10} \underbrace{200\, e^{0.01t}}_{R(t)}\, \underbrace{e^{-0.015(10-t)}}_{S(T-t)}\, dt$$

$$\approx 516{,}425 + \int_0^{10} 200e^{0.01t}\,[e^{-0.015(10)}\, e^{0.015t}]\, dt \qquad \text{since } e^{a-b} = e^a\, e^{-b}$$

$$\approx 516{,}425 + 200e^{-0.015(10)} \int_0^{10} [e^{0.01t}\, e^{0.015t}]\, dt \qquad \text{factor } 200e^{-0.015(10)} \text{ outside the integral}$$

$$\approx 516{,}425 + 172.14 \int_0^{10} e^{0.025t}\, dt \qquad \text{since } e^{a+b} = e^a\, e^b \text{ and } 200e^{-0.015(10)} \approx 172.14$$

$$\approx 516{,}425 + 172.14 \left[\frac{e^{0.025t}}{0.025}\right]\Bigg|_0^{10} \qquad \text{exponential rule for integration}$$

$$\approx 516{,}425 + \frac{172.14}{0.025}\,[e^{0.025(10)} - e^0]$$

$$\approx 518{,}381$$

Thus, the population of the colony declines from 600,000 to about 518,381 during the first 10 hours after the toxin is introduced.

## The Flow of Blood through an Artery

Biologists have found that the speed of blood in an artery is a function of the distance of the blood from the artery's central axis. According to Poiseuille's law, the speed (in centimeters per second) of blood that is $r$ centimeters from the central axis of the artery is $S(r) = k(R^2 - r^2)$, where $R$ is the radius of the artery and $k$ is a constant. In Example 5.6.3, you will see how to use this information to compute the rate at which blood flows through the artery.

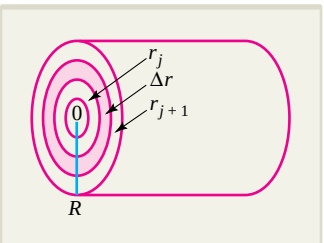



**FIGURE 5.24** Subdividing a cross section of an artery into concentric rings.

### EXAMPLE 5.6.3

Find an expression for the rate (in cubic centimeters per second) at which blood flows through an artery of radius $R$ if the speed of the blood $r$ centimeters from the central axis is $S(r) = k(R^2 - r^2)$, where $k$ is a constant.

### Solution

To approximate the volume of blood that flows through a cross section of the artery each second, divide the interval $0 \le r \le R$ into $n$ equal subintervals of width $\Delta r$ centimeters and let $r_j$ denote the beginning of the $j$th subinterval. These subintervals determine $n$ concentric rings as illustrated in Figure 5.24.

If $\Delta r$ is small, the area of the $j$th ring is approximately the area of a rectangle whose length is the circumference of the (inner) boundary of the ring and whose width is $\Delta r$. That is,

$$\text{Area of } j\text{th ring} \approx 2\pi r_j \Delta r$$

If you multiply the area of the $j$th ring (square centimeters) by the speed (centimeters per second) of the blood flowing through this ring, you get the volume rate (cubic centimeters per second) at which blood flows through the $j$th ring. Since the speed of blood flowing through the $j$th ring is approximately $S(r_j)$ centimeters per second, it follows that

$$\begin{aligned}\begin{pmatrix}\text{Volume rate of flow}\\ \text{through } j\text{th ring}\end{pmatrix} &\approx \begin{pmatrix}\text{area of}\\ j\text{th ring}\end{pmatrix}\begin{pmatrix}\text{speed of blood}\\ \text{through } j\text{th ring}\end{pmatrix}\\ &\approx (2\pi r_j \Delta r)S(r_j)\\ &\approx (2\pi r_j \Delta r)[k(R^2 - r_j^2)]\\ &\approx 2\pi k(R^2 r_j - r_j^3)\Delta r\end{aligned}$$

The volume rate of flow of blood through the entire cross section is the sum of $n$ such terms, one for each of the $n$ concentric rings. That is,

$$\text{Volume rate of flow} \approx \sum_{j=1}^{n} 2\pi k(R^2 r_j - r_j^3)\Delta r$$

As $n$ increases without bound, this approximation approaches the true value of the rate of flow. In other words,

$$\begin{aligned}\text{Volume rate of flow} &= \lim_{n\to+\infty} \sum_{j=1}^{n} 2\pi k(R^2 r_j - r_j^3)\Delta r\\ &= \int_0^R 2\pi k(R^2 r - r^3)\,dr\\ &= 2\pi k\left(\frac{R^2}{2}r^2 - \frac{1}{4}r^4\right)\Bigg|_0^R\\ &= \frac{\pi k R^4}{2}\end{aligned}$$

Thus, the blood is flowing at the rate of $\dfrac{\pi k R^4}{2}$ cubic centimeters per second.

## Cardiac Output

In studying the cardiovascular system, physicians and medical researchers are often interested in knowing the **cardiac output** of a person's heart, which is the volume of blood it pumps in unit time. Cardiac output is measured by a procedure called the **dye dilution method.*** A known amount of dye is injected into a vein near the heart. The dye then circulates with the blood through the right side of the heart, the

*See the module, "Measuring Cardiac Output," by B. Horelick and S. Koont, *UMAP Modules 1977: Tools for Teaching*, Lexington, MA: Consortium for Mathematics and Its Applications, Inc., 1978. Another good source is *Calculus and Its Applications*, by S. Farlow and G. Haggard, Boston: McGraw-Hill, 1990, pp. 332–334.

lungs, and the left side of the heart before finally appearing in the arterial system. A monitoring probe is introduced into the aorta, and blood samples are taken at regular time intervals to measure the concentration of dye leaving the heart until all the dye has passed the monitoring point. A typical concentration graph is shown in Figure 5.25.

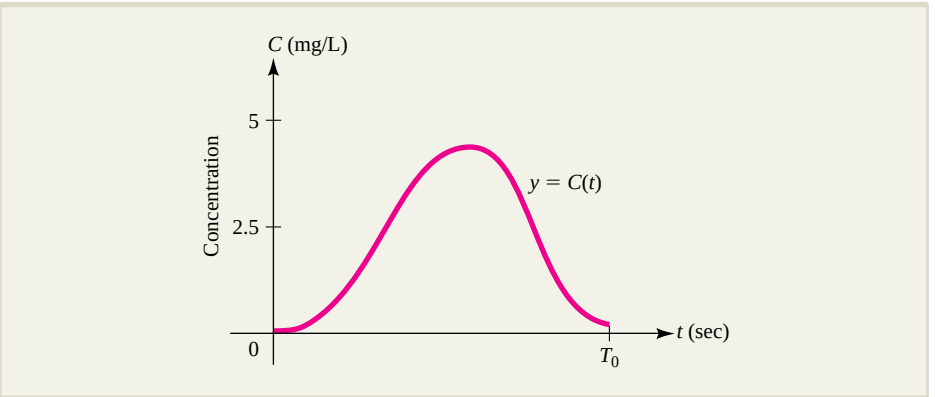



**FIGURE 5.25** A typical graph showing concentration of dye in a patient's aorta.

Suppose the amount of dye injected is $D$ mg, and that $C(t)$ (mg/L) is the concentration of dye at time $t$. Let $T_0$ denote the total time required for all the dye to pass the monitoring point, and divide the time interval $0 \le t \le T_0$ into $n$ equal subintervals, each of length $\Delta t = \dfrac{T_0 - 0}{n}$. If $R$ is the cardiac output (liters/min), then approximately $R\Delta t$ liters of blood flow past the monitoring probe during the $k$th time subinterval $t_{k-1} \le t \le t_k$, carrying $C(t_k)R\Delta t$ mg of dye. Adding up the amounts of dye over all $n$ subintervals, we obtain the sum

$$\sum_{k=1}^{n} C(t_k)R\Delta t$$

as an approximation for the total amount of dye, and by taking the limit as $n \to +\infty$ we obtain the actual total amount of dye as a definite integral:

$$\lim_{n\to+\infty} \sum_{k=1}^{n} C(t_k)R\Delta t = \int_0^{T_0} C(t)R\,dt = R\int_0^{T_0} C(t)\,dt$$

Since $D$ milligrams of dye were originally injected, we must have

$$D = R\int_0^{T_0} C(t)\,dt$$

so the cardiac output is given by

$$R = \frac{D}{\displaystyle\int_0^{T_0} C(t)\,dt}$$

## EXAMPLE 5.6.4

A physician injects 4 mg of dye into a vein near the heart of a patient, and a monitoring device records the concentration of dye in the blood at regular intervals over a 23-second period. It is determined that the concentration at time $t$ $(0 \le t \le 23)$ is closely approximated by the function

$$C(t) = 0.09t^2e^{-0.0007t^3} \quad \text{mg/L}$$

Based on this information, what is the patient's cardiac output?

**EXPLORE!**

Refer to Example 5.6.4. Graph the dye concentration function $C(t) = 0.09t^2e^{-0.0007t^3}$ using the window [0, 23.5]5 by [−1, 6]1. Compute the patient's cardiac output assuming only a 20-second observation period and compare with the results obtained in the example (for a 23-second period).

### Solution

Integrating $C(t)$ over the time interval $0 \le t \le 23$, we find that

$$\begin{aligned}
\int_0^{23} C(t)\,dt &= \int_0^{23} 0.09t^2e^{-0.0007t^3}\,dt \\
&= 0.09\int_0^{23} e^{-0.0007t^3}(t^2\,dt) && \text{substitute } u = t^3,\ du = 3t^2\,dt \\
&= 0.09\int_0^{12{,}167} e^{-0.0007u}\left(\frac{1}{3}\,du\right) && \text{when } x = 0,\ u = 0;\ \text{when } x = 23,\ u = (23)^3 = 12{,}167 \\
&= \frac{0.09}{3}\left(\frac{e^{-0.0007u}}{-0.0007}\right)\Bigg|_0^{12{,}167} \\
&\approx -42.86\left[e^{-0.0007(12{,}167)} - e^0\right] \\
&\approx 42.85
\end{aligned}$$

Thus, the cardiac output is

$$\begin{aligned}
R &= \frac{4}{\displaystyle\int_0^{23} C(t)\,dt} \\
&\approx \frac{4}{42.85} \approx 0.093 \quad \text{liters/sec}
\end{aligned}$$

or equivalently,

$$R \approx (0.093 \text{ liters/sec})(60 \text{ sec/min}) \approx 5.6 \text{ liters/min}$$

## Population Density

The **population density** of an urban area is the number of people $p(r)$ per square mile that are located a distance $r$ miles from the city center. We can determine the total population $P$ of the portion of the city that lies within $R$ miles of the city center by using integration.

Our approach to using population density to determine total population will be similar to the approach used earlier in this section to determine the flow of blood through an artery. In particular, divide the interval $0 \le r \le R$ into $n$ subintervals, each of width $\Delta r = \dfrac{R}{n}$, and let $r_k$ denote the beginning (left endpoint) of the $k$th subinterval, for $k = 1, 2, \ldots, n$. These subintervals determine $n$ concentric rings, centered on the city center as shown in Figure 5.26.

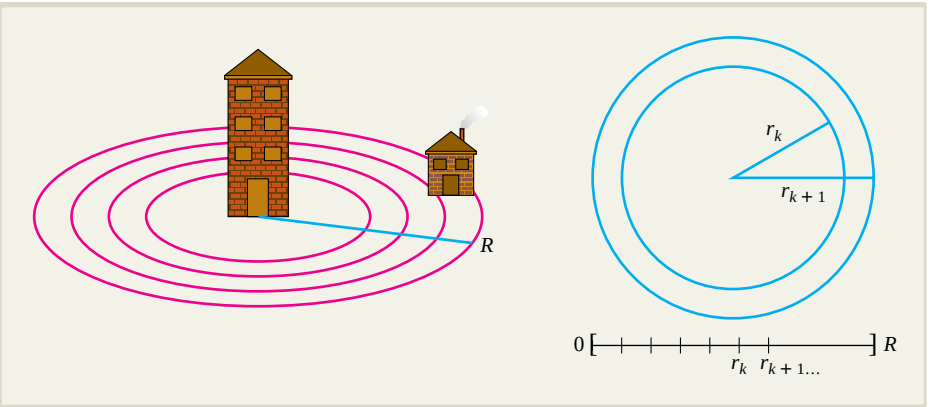



**FIGURE 5.26** Subdividing an urban area into concentric rings.

The area of the $k$th ring is approximately the area of a rectangle whose length is the circumference of the inner boundary of the ring and whose width is $\Delta r$. That is,

$$\text{Area of } k\text{th ring} \approx 2\pi r_k \Delta r$$

and since the population density is $p(r)$ people per square mile, it follows that

$$\text{Population within } k\text{th ring} \approx \underbrace{p(r_k)}_{\substack{\text{population}\\ \text{per unit area}}} \cdot \underbrace{[2\pi r_k \Delta r]}_{\substack{\text{area}\\ \text{of ring}}} = 2\pi r_k\, p(r_k)\, \Delta r$$

We can estimate the total area with the bounding radius $R$ by adding up the populations within the approximating rings; that is, by the Riemann sum

$$\begin{bmatrix}\text{Total population}\\ \text{within radius } R\end{bmatrix} = P(R) \approx \sum_{k=1}^{n} 2\pi r_k\, p(r_k)\, \Delta r$$

By taking the limit as $n \to \infty$, the estimate approaches the true value of the total population $P$, and since the limit of a Riemann sum is a definite integral, we have

$$P(R) = \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi\, r_k\, p(r_k)\, \Delta r = \int_0^R 2\pi r p(r)\, dr$$

To summarize:

**Total Population from Population Density** ■ If a concentration of individuals has population density $p(r)$ individuals per square unit at a distance $r$ from the center of concentration, then the total population $P(R)$ located within distance $R$ from the center is given by

$$P(R) = \int_0^R 2\pi r p(r)\, dr$$

**NOTE** We found it convenient to derive the population density formula by considering the population of a city. However, the formula also applies to more general population concentrations, such as bacterial colonies or even the "population" of water drops from a sprinkler system. ■

## EXAMPLE 5.6.5

A city has population density $p(r) = 3e^{-0.01r^2}$, where $p(r)$ is the number of people (in thousands) per square mile at a distance of $r$ miles from the city center.

**a.** What population lives within a 5-mile radius of the city center?

**b.** The city limits are set at a radius $R$ where the population density is 1,000 people per square mile. What is the total population within the city limits?

### Solution

**a.** The population within a 5-mile radius is

$$P(5) = \int_0^5 2\pi r\,(3e^{-0.01r^2})\,dr = 6\pi \int_0^5 e^{-0.01r^2} r\,dr$$

Using the substitution $u = -0.01\,r^2$, we find that

$$du = -0.01(2r\,dr) \quad \text{or} \quad r\,dr = \frac{du}{-0.02} = -50\,du$$

In addition, the limits of integration are transformed as follows:

$$\text{If } r = 5, \text{ then } u = -0.01(5)^2 = -0.25$$
$$\text{If } r = 0, \text{ then } u = -0.01(0)^2 = 0$$

Therefore, we have

$$\begin{aligned} P(5) &= 6\pi \int_0^5 e^{-0.01r^2} r\,dr \\ &= 6\pi \int_0^{-0.25} e^u(-50\,du) \qquad \text{since } r\,dr = -50\,du \\ &= 6\pi\,(-50)[e^u]\Big|_{u=0}^{u=-0.25} \\ &= -300\pi[e^{-0.25} - e^0] \\ &\approx 208.5 \end{aligned}$$

So roughly 208,500 people live within a 5-mile radius of the city center.

**b.** To find the radius $R$ that corresponds to the city limits, we want the population density to be 1 (one thousand), so we solve

$$\begin{aligned} 3e^{-0.01R^2} &= 1 \\ e^{-0.01R^2} &= \frac{1}{3} \\ -0.01R^2 &= \ln\left(\frac{1}{3}\right) \qquad \text{take logarithms on both sides} \\ R^2 &= \frac{\ln\left(\frac{1}{3}\right)}{-0.01} = 109.86 \\ R &\approx 10.48 \end{aligned}$$

Finally, using the substitution $u = -0.01r^2$ from part (a), we find that the population within the city limits is

$$\begin{aligned} P(10.48) &= 6\pi \int_0^{10.48} e^{-0.01r^2} r\, dr \\ &= 6\pi \int_0^{-1.1} e^u(-50\, du) \\ &\approx -300\pi\, [e^u]\Big|_{u=0}^{u=-1.1} \\ &\approx -300\pi[e^{-1.1} - e^0] \\ &\approx 628.75 \end{aligned}$$

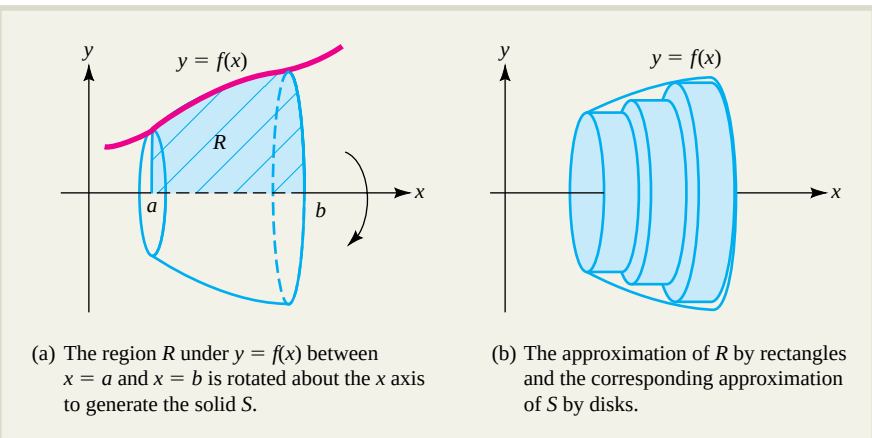
Limits of integration:
when $r = 10.48$, then $u = -0.01(10.48)^2 \approx -1.1$
when $r = 0$, then $u = 0$

Thus, approximately 628,750 people live within the city limits.

## The Volume of a Solid of Revolution

In the next application, the definite integral is used to find the volume of a **solid of revolution** formed by revolving a region $R$ in the $xy$ plane about the $x$ axis.

The technique is to express the volume of the solid as the limit of a sum of the volumes of approximating disks. In particular, suppose that $S$ is the solid formed by rotating the region $R$ under the curve $y = f(x)$ between $x = a$ and $x = b$ about the $x$ axis, as shown in Figure 5.27a. Divide the interval $a \le x \le b$ into $n$ equal subintervals of length $\Delta x$. Then approximate the region $R$ by $n$ rectangles and the solid $S$ by the corresponding $n$ cylindrical disks formed by rotating these rectangles about the $x$ axis. The general approximation procedure is illustrated in Figure 5.27b for the case where $n = 3$.




(a) The region $R$ under $y = f(x)$ between $x = a$ and $x = b$ is rotated about the $x$ axis to generate the solid $S$.

(b) The approximation of $R$ by rectangles and the corresponding approximation of $S$ by disks.

**FIGURE 5.27** A solid $S$ formed by rotating the region $R$ about the $x$ axis.

If $x_j$ denotes the beginning (left endpoint) of the $j$th subinterval, then the $j$th rectangle has height $f(x_j)$ and width $\Delta x$ as shown in Figure 5.28a. The $j$th

approximating disk formed by rotating this rectangle about the $x$ axis is shown in Figure 5.28b.

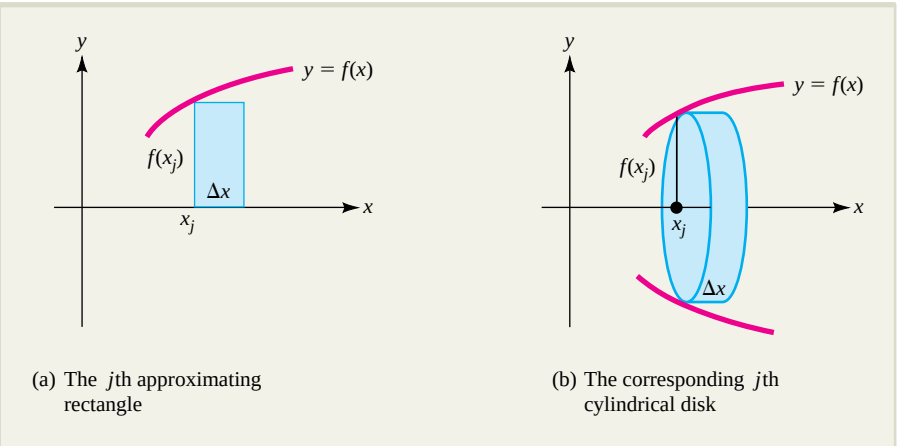



**FIGURE 5.28** The volume of the solid $S$ is approximated by adding volumes of approximating disks.

Since the $j$th approximating cylindrical disk has radius $r_j = f(x_j)$ and thickness $\Delta x$, its volume is

$$\begin{aligned}\text{Volume of } j\text{th disk} &= (\text{area of circular cross section})(\text{width})\\ &= \pi r_j^2(\text{width}) = \pi[f(x_j)]^2\,\Delta x\end{aligned}$$

The total volume of $S$ is approximately the sum of the volumes of the $n$ disks; that is,

$$\text{Volume of } S \approx \sum_{j=1}^{n} \pi[f(x_j)]^2\,\Delta x$$

The approximation improves as $n$ increases and

$$\text{Volume of } S = \lim_{n\to\infty} \sum_{j=1}^{n} \pi[f(x_j)]^2\,\Delta x = \pi\int_a^b [f(x)]^2\,dx$$

To summarize:

### Volume Formula

Suppose $f(x)$ is continuous and $f(x) \geq 0$ on $a \leq x \leq b$ and let $R$ be the region under the curve $y = f(x)$ between $x = a$ and $x = b$. Then the solid $S$ formed by rotating $R$ about the $x$ axis has volume

$$\text{Volume of } S = \pi\int_a^b [f(x)]^2\,dx$$

Here are two examples.

## EXAMPLE 5.6.6

Find the volume of the solid $S$ formed by revolving the region under the curve $y = x^2 + 1$ from $x = 0$ to $x = 2$ about the $x$ axis.

### Solution

The region, the solid of revolution, and the $j$th disk are shown in Figure 5.29.

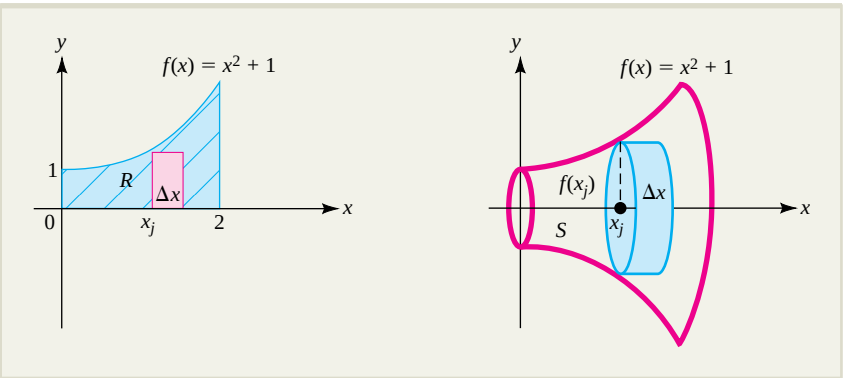



**FIGURE 5.29** The solid formed by rotating the region under the curve $y = x^2 + 1$ between $x = 0$ and $x = 2$ about the $x$ axis.

The radius of the $j$th disk is $f(x_j) = x_j^2 + 1$. Hence,

$$\text{Volume of } j\text{th disk} = \pi[f(x_j)]^2\,\Delta x = \pi(x_j^2 + 1)^2\,\Delta x$$

and

$$\begin{aligned}\text{Volume of } S &= \lim_{n\to\infty}\sum_{j=1}^{n}\pi(x_j^2 + 1)^2\,\Delta x\\ &= \pi\int_0^2 (x^2 + 1)^2\,dx\\ &= \pi\int_0^2 (x^4 + 2x^2 + 1)\,dx\\ &= \pi\left(\frac{1}{5}x^5 + \frac{2}{3}x^3 + x\right)\Bigg|_0^2 = \frac{206}{15}\pi \approx 43.14\end{aligned}$$

## EXAMPLE 5.6.7

A tumor has approximately the same shape as the solid formed by rotating the region under the curve $y = \frac{1}{3}\sqrt{16 - 4x^2}$ about the $x$ axis, where $x$ and $y$ are measured in centimeters. Find the volume of the tumor.

## Solution

The curve intersects the $x$ axis where $y = 0$; that is, where

$$\begin{aligned} \frac{1}{3}\sqrt{16 - 4x^2} &= 0 \\ 16 &= 4x^2 && \text{since } \sqrt{a - b} = 0 \text{ only if } a = b \\ x^2 &= 4 && \text{divide both sides by 4} \\ x &= \pm 2 \end{aligned}$$

The curve (called an *ellipse*) is shown in Figure 5.30.

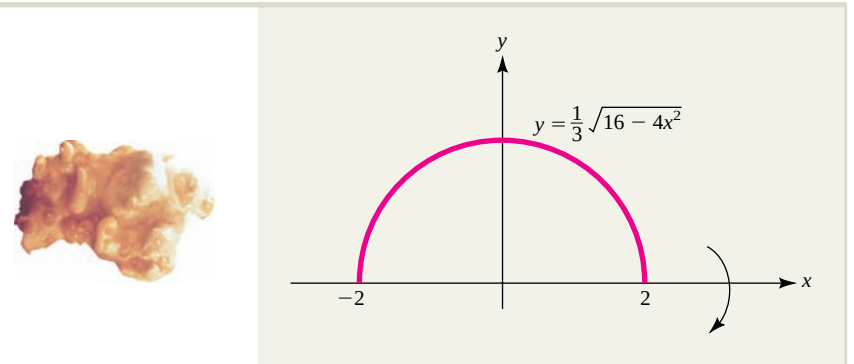



**FIGURE 5.30** Tumor with the approximate shape of the solid formed by rotating the curve $y = \frac{1}{3}\sqrt{16 - 4x^2}$ about the $x$ axis.

Let $f(x) = \frac{1}{3}\sqrt{16 - 4x^2}$. Then the volume of the solid of revolution is given by

$$\begin{aligned} V &= \int_{-2}^{2} \pi[f(x)]^2\,dx = \int_{-2}^{2} \pi\left[\frac{1}{3}\sqrt{16 - 4x^2}\right]^2 dx \\ &= \int_{-2}^{2} \frac{\pi}{9}(16 - 4x^2)\,dx \\ &= \frac{\pi}{9}\left[16x - \frac{4}{3}x^3\right]\Bigg|_{-2}^{2} \\ &= \frac{\pi}{9}\left[16(2) - \frac{4}{3}(2)^3\right] - \frac{\pi}{9}\left[16(-2) - \frac{4}{3}(-2)^3\right] \\ &\approx 14.89 \end{aligned}$$

Thus, the volume of the tumor is approximately 15 $\text{cm}^3$.

# EXERCISES ■ 5.6

**SURVIVAL AND RENEWAL** *In Exercises 1 through 6, an initial population $P_0$ is given along with a renewal rate R, and a survival function S(t). In each case, use the given information to find the population at the end of the indicated term T.*

1. $P_0 = 50{,}000$; $R(t) = 40$; $S(t) = e^{-0.1t}$, $t$ in months; term $T = 5$ months
2. $P_0 = 100{,}000$; $R(t) = 300$; $S(t) = e^{-0.02t}$, $t$ in days; term $T = 10$ days
3. $P_0 = 500{,}000$; $R(t) = 800$; $S(t) = e^{-0.011t}$, $t$ in years; term $T = 3$ years
4. $P_0 = 800{,}000$; $R(t) = 500$; $S(t) = e^{-0.005t}$, $t$ in months; term $T = 5$ months
5. $P_0 = 500{,}000$; $R(t) = 100e^{0.01t}$; $S(t) = e^{-0.013t}$, $t$ in years; term $T = 8$ years
6. $P_0 = 300{,}000$; $R(t) = 150e^{0.012t}$; $S(t) = e^{-0.02t}$, $t$ in months; term $T = 20$ months

**VOLUME OF A SOLID OF REVOLUTION** *In Exercises 7 through 14, find the volume of the solid of revolution formed by rotating the region R about the x axis.*

7. $R$ is the region under the line $y = 3x + 1$ from $x = 0$ to $x = 1$.
8. $R$ is the region under the curve $y = \sqrt{x}$ from $x = 1$ to $x = 4$.
9. $R$ is the region under the curve $y = x^2 + 2$ from $x = -1$ to $x = 3$.
10. $R$ is the region under the curve $y = 4 - x^2$ from $x = -2$ to $x = 2$.
11. $R$ is the region under the curve $y = \sqrt{4 - x^2}$ from $x = -2$ to $x = 2$.
12. $R$ is the region under the curve $y = \dfrac{1}{x}$ from $x = 1$ to $x = 10$.
13. $R$ is the region under the curve $y = \dfrac{1}{\sqrt{x}}$ from $x = 1$ to $x = e^2$.
14. $R$ is the region under the curve $y = e^{-0.1x}$ from $x = 0$ to $x = 10$.
15. **NET POPULATION GROWTH** It is projected that $t$ years from now the population of a certain country will be changing at the rate of $e^{0.02t}$ million per year. If the current population is 50 million, what will be the population 10 years from now?
16. **NET POPULATION GROWTH** A study indicates that $x$ months from now, the population of a certain town will be increasing at the rate of $10 + 2\sqrt{x}$ people per month. By how much will the population increase over the next 9 months?
17. **GROUP MEMBERSHIP** A national consumers' association has compiled statistics suggesting that the fraction of its members who are still active $t$ months after joining is given by $f(t) = e^{-0.2t}$. A new local chapter has 200 charter members and expects to attract new members at the rate of 10 per month. How many members can the chapter expect to have at the end of 8 months?
18. **POLITICAL TRENDS** Sarah Greene is running for mayor. Polls indicate that the fraction of those who support her $t$ weeks after first learning of her candidacy is given by $f(t) = e^{-0.03t}$. At the time she declared her candidacy, 25,000 people supported her, and new converts are being added at the constant rate of 100 people per week. Approximately how many people are likely to vote for her if the election is held 20 weeks from the day she entered the race?
19. **SPREAD OF DISEASE** A new strain of influenza has just been declared an epidemic by health officials. Currently, 5,000 people have the disease and 60 more victims are added each day. If the fraction of infected people who still have the disease $t$ days after contracting it is given by $f(t) = e^{-0.02t}$, how many people will have the flu 30 days from now?
20. **NUCLEAR WASTE** A certain nuclear power plant produces radioactive waste in the form of strontium-90 at the constant rate of 500 pounds per year. The waste decays exponentially with a half-life of 28 years. How much of the radioactive waste from the nuclear plant will be present after 140 years? [*Hint:* Think of this as a survival and renewal problem.]
21. **ENERGY CONSUMPTION** The administration of a small country estimates that the demand for oil is increasing exponentially at the rate of 10%

per year. If the demand is currently 30 billion barrels per year, how much oil will be consumed in this country during the next 10 years?

**22.** **POPULATION GROWTH** The administrators of a town estimate that the fraction of people who will still be residing in the town $t$ years after they arrive is given by the function $f(t) = e^{-0.04t}$. If the current population is 20,000 people and new townspeople arrive at the rate of 500 per year, what will be the population 10 years from now?

**23.** **COMPUTER DATING** The operators of a new computer dating service estimate that the fraction of people who will retain their membership in the service for at least $t$ months is given by the function $f(t) = e^{-t/10}$. There are 8,000 charter members, and the operators expect to attract 200 new members per month. How many members will the service have 10 months from now?

**24.** **FLOW OF BLOOD** Calculate the rate (in cubic centimeters per second) at which blood flows through an artery of radius 0.1 centimeter if the speed of the blood $r$ centimeters from the central axis is $8 - 800r^2$ centimeters per second.

**25.** **CARDIAC OUTPUT** A physician injects 5 mg of dye into a vein near the heart of a patient and by monitoring the concentration of dye in the blood over a 24-second period, determines that the concentration of dye leaving the heart after $t$ seconds ($0 \le t \le 24$) is given by the function

$$C(t) = -0.028t^2 + 0.672t \quad \text{mg/L}$$

**a.** Use this information to find the patient's cardiac output.

**b.** Sketch the graph of $C(t)$, and compare it to the graph in Figure 5.25. How are the two graphs alike? How are they different?

**26.** **CARDIAC OUTPUT** Answer the questions in Exercise 25 for the dye concentration function

$$C(t) = \begin{cases} 0 & \text{for } 0 \le t \le 2 \\ -0.034(t^2 - 26t + 48) & \text{for } 2 \le t \le 24 \end{cases}$$

**27.** **CARDIAC OUTPUT** Answer the questions in Exercise 25 for the dye concentration function

$$C(t) = \frac{1}{12{,}312}(t^4 - 48t^3 + 378t^2 + 4{,}752t)$$

**28.** **POPULATION DENSITY** The population density $r$ miles from the center of a certain city is $D(r) = 5{,}000(1 + 0.5r^2)^{-1}$ people per square mile.

**a.** How many people live within 5 miles of the city center?

**b.** The city limits are set at a radius $L$ where the population density is 1,000 people per square mile. What is $L$ and what is the total population within the city limits?

**29.** **POPULATION DENSITY** The population density $r$ miles from the center of a certain city is $D(r) = 25{,}000e^{-0.05r^2}$ people per square mile. How many people live between 1 and 2 miles from the city center?

**30.** **POISEUILLE'S LAW** Blood flows through an artery of radius $R$. At a distance $r$ centimeters from the central axis of the artery, the speed of the blood is given by $S(r) = k(R^2 - r^2)$. Show that the average velocity of the blood is one-half the maximum speed.

**31.** **CHOLESTEROL REGULATION** Fat travels through the bloodstream attached to protein in a combination called a *lipoprotein*. Low-density lipoprotein (LDL) picks up cholesterol from the liver and delivers it to the cells, dropping off any excess cholesterol on the artery walls. Too much LDL in the bloodstream increases the risk of heart disease and stroke. A patient with a high level of LDL receives a drug that is found to reduce the level at a rate given by

$$L'(t) = -0.3t(49 - t^2)^{0.4} \quad \text{units/day}$$

where $t$ is the number of days after the drug is administered, for $0 \le t \le 7$.

**a.** By how much does the patient's LDL level change during the first 3 days after the drug is administered?

**b.** Suppose the patient's LDL level is 120 at the time the drug is administered. Find $L(t)$.

**c.** The recommended "safe" LDL level is 100. How many days does it take for the patient's LDL level to be "safe"?

**32.** **CHOLESTEROL REGULATION** During his annual medical checkup, a man is advised by his doctor to adopt a regimen of exercise, diet, and medication to lower his blood cholesterol level to 220 milligrams per deciliter (mg/dL). Suppose the man finds that his cholesterol level $t$ days after beginning the regimen is

$$L(t) = 190 + 65e^{-0.003t}$$

**a.** What is the man's cholesterol level when he begins the regimen?

**b.** How many days $N$ must the man remain on the regimen to lower his cholesterol level to 220 mg/dL?

**c.** What was the man's average cholesterol level during the first 30 days of the regimen? What was the average level over the entire period $0 \le t \le N$ of the regimen?

**33.** **BACTERIAL GROWTH** An experiment is conducted with two bacterial colonies, each of which initially has a population of 100,000. In the first colony, a mild toxin is introduced that restricts growth so that only 50 new individuals are added per day and the fraction of individuals that survive at least $t$ days is given by $f(t) = e^{-0.011t}$. The growth of the second colony is restricted indirectly, by limiting food supply and space for expansion, and after $t$ days, it is found that this colony contains

$$P(t) = \frac{5{,}000}{1 + 49e^{0.009t}}$$

thousand individuals. Which colony is larger after 50 days? After 100 days? After 300 days?

**34.** **GROUP MEMBERSHIP** A group has just been formed with an initial membership of 10,000. Suppose that the fraction of the membership of the group that remain members for at least $t$ years after joining is $S(t) = e^{-0.03t}$, and that at time $t$, new members are being added at the rate of $R(t) = 10e^{0.017t}$ members per year. How many members will the group have 5 years from now?

**35.** **GROWTH OF AN ENDANGERED SPECIES** Environmentalists estimate that the population of a certain endangered species is currently 3,000. The population is expected to be growing at the rate of $R(t) = 10e^{0.01t}$ individuals per year $t$ years from now, and the fraction that survive $t$ years is given by $S(t) = e^{-0.07t}$. What will the population of the species be in 10 years?

**36.** **POPULATION TRENDS** The population of a small town is currently 85,000. A study commissioned by the mayor's office finds that people are settling in the town at the rate of $R(t) = 1{,}200e^{0.01t}$ per year and that the fraction of the population who continue to live in the town $t$ years after arriving is given by $S(t) = e^{-0.02t}$. How many people will live in the town in 10 years?

**37.** **POPULATION TRENDS** Answer the question in Exercise 36 for a constant renewal rate $R = 1{,}000$ and the survival function

$$S(t) = \frac{1}{t + 1}$$

**38.** **EVALUATING DRUG EFFECTIVENESS** A pharmaceutical firm has been granted permission by the FDA to test the effectiveness of a new drug in combating a virus. The firm administers the drug to a test group of uninfected but susceptible individuals, and using statistical methods, determines that $t$ months after the test begins, people in the group are becoming infected at the rate of $D'(t)$ hundred individuals per month, where

$$D'(t) = 0.2 - 0.04t^{1/4}$$

Government figures indicate that without the drug, the infection rate would have been $W'(t)$ hundred individuals per month, where

$$W'(t) = \frac{0.8e^{0.13t}}{(1 + e^{0.13t})^2}$$

If the test is evaluated 1 year after it begins, how many people does the drug protect from infection? What percentage of the people who would have been infected if the drug had not been used were protected from infection by the drug?

**39.** **EVALUATING DRUG EFFECTIVENESS** Repeat Exercise 38 for another drug for which the infection rate is

$$D'(t) = 0.12 + \frac{0.08}{t + 1}$$

Assume the government comparison rate stays the same; that is,

$$W'(t) = \frac{0.8e^{0.13t}}{(1 + e^{0.13t})^2}$$

**40.** **LIFE EXPECTANCY** In a certain undeveloped country, the life expectancy of a person $t$ years old is $L(t)$ years, where

$$L(t) = 41.6(1 + 1.07t)^{0.13}$$

**a.** What is the life expectancy of a person in this country at birth? At age 50?

**b.** What is the average life expectancy of all people in this country between the ages of 10 and 70?

**c.** Find the age $T$ such that $L(T) = T$. Call $T$ the *life limit* for this country. What can be said about the life expectancy of a person older than $T$ years?

**d.** Find the average life expectancy $L_e$ over the age interval $0 \le t \le T$. Why is it reasonable to call $L_e$ the *expected length of life* for people in this country?

**41. LIFE EXPECTANCY** Answer the questions in Exercise 40 for a country whose life expectancy function is

$$L(t) = \frac{110e^{0.015t}}{1 + e^{0.015t}}$$

**42. ENERGY EXPENDED IN FLIGHT** In an investigation by V. A. Tucker and K. Schmidt-Koenig,* it was determined that the energy $E$ expended by a bird in flight varies with the speed $v$ (km/hr) of the bird. For a particular kind of parakeet, the energy expenditure changes at a rate given by

$$\frac{dE}{dv} = \frac{0.31v^2 - 471.75}{v^2} \quad \text{for } v > 0$$

where $E$ is given in joules per gram mass per kilometer. Observations indicate that the parakeet tends to fly at the speed $v_{\min}$ that minimizes $E$.

**a.** What is the most economical speed $v_{\min}$?

**b.** Suppose that when the parakeet flies at the most economical speed $v_{\min}$ its energy expenditure is $E_{\min}$. Use this information to find $E(v)$ for $v > 0$ in terms of $E_{\min}$.

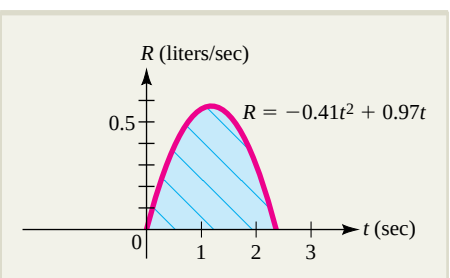
**43. MEASURING RESPIRATION** A pneumotachograph is a device used by physicians to graph the rate of air flow into and out of the lungs as a patient breathes. The graph in the accompanying figure shows the rate of inspiration (breathing in) for a particular patient. The area under the graph measures the total volume of air inhaled by the patient during the inspiration phase of one breathing cycle. Assume the inspiration rate is given by

$$R(t) = -0.41t^2 + 0.97t \quad \text{liters/sec}$$

**a.** How long is the inspiration phase?

**b.** Find the volume of air taken into the patient's lungs during the inspiration phase.

**c.** What is the average flow rate of air into the lungs during the inspiration phase?




EXERCISE 43

*E. Batschelet, *Introduction to Mathematics for Life Scientists*, 3rd ed., New York, Springer-Verlag, 1979, p. 299.

**44. MEASURING RESPIRATION** Repeat Exercise 43 with the inspiration rate function

$$R(t) = -1.2t^3 + 5.72t \quad \text{liters/sec}$$

and sketch the graph of $R(t)$.

**45. WATER POLLUTION** A ruptured pipe in an offshore oil rig produces a circular oil slick that is $T$ feet thick at a distance $r$ feet from the rupture, where

$$T(r) = \frac{3}{2 + r}$$

At the time the spill is contained, the radius of the slick is 7 feet. We wish to find the volume of oil that has been spilled.

**a.** Sketch the graph of $T(r)$. Notice that the volume we want is obtained by rotating the curve $T(r)$ about the $T$ *axis* (vertical axis) rather than the $r$ axis (horizontal axis).

**b.** Solve the equation $T = \dfrac{3}{2 + r}$ for $r$ in terms of $T$. Sketch the graph of $r(T)$, with $T$ on the horizontal axis.

**c.** Find the required volume by rotating the graph of $r(T)$ found in part (b) about the $T$ axis.

**46. WATER POLLUTION** Rework Exercise 45 for a situation with spill thickness

$$T(r) = \frac{2}{1 + r^2}$$

($T$ and $r$ in feet) and radius of containment 9 feet.

**47. AIR POLLUTION** Particulate matter emitted from a smokestack is distributed in such a way that $r$ miles from the stack, the pollution density is $p(r)$ units per square mile, where

$$p(r) = \frac{200}{5 + 2r^2}$$

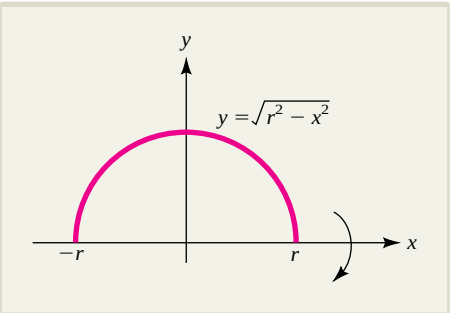
**a.** What is the total amount of pollution within a 3-mile radius of the smokestack?

**b.** Suppose a health agency determines that it is unsafe to live within a radius $L$ of the smokestack where the pollution density is at least four units per square mile. What is $L$, and what is the total amount of pollution in the unsafe zone?

**48. VOLUME OF A SPHERE** Use integration to show that a sphere of radius $r$ has volume

$$V = \frac{4}{3}\pi r^3$$

[*Hint:* Think of the sphere as the solid formed by rotating the region under the semicircle shown in the accompanying figure about the $x$ axis.]

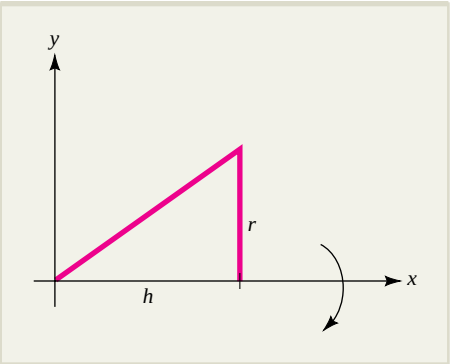



EXERCISE 48

**49.** VOLUME OF A CONE Use integration to show that a right circular cone of height $h$ and top radius $r$ has volume

$$V = \frac{1}{3}\pi r^2 h$$

[*Hint:* Think of the cone as a solid formed by rotating the triangle shown in the accompanying figure about the $x$ axis.]




EXERCISE 49

CHAPTER SUMMARY

## Important Terms, Symbols, and Formulas

Antiderivative; indefinite integral: (372, 374)

$$\int f(x)\,dx = F(x) + C \text{ if and only if } F'(x) = f(x)$$

Power rule: (375)

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

Logarithmic rule: $\int \frac{1}{x}\,dx = \ln|x| + C$ (375)

Exponential rule: $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$ (375)

Constant multiple rule: (376)

$$\int kf(x)\,dx = k\int f(x)dx$$

Sum rule: (376)

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Initial value problem (378)

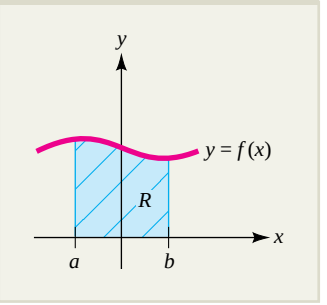
Integration by substitution: (386)

$$\int g(u(x))u'(x)\,dx = \int g(u)\,du \quad \text{where } u = u(x),\ du = u'(x)\,dx$$

Definite integral: (401)

$$\int_a^b f(x)\,dx = \lim_{n\to+\infty}[f(x_1) + \cdots + f(x_n)]\Delta x$$

Area under a curve: (399, 401)




Area of $R$

$$= \int_a^b f(x)\,dx$$

Special rules for definite integrals: (404)

$$\int_a^a f(x)\,dx = 0$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

Constant multiple rule: (404)

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx \quad \text{for constant } k$$

Sum rule: (404)

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Difference rule: (404)

$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Subdivision rule: (404)

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

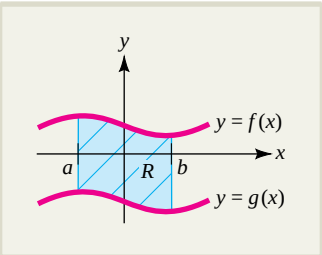
Fundamental theorem of calculus: (402)

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{where } F'(x) = f(x)$$

Net change of $Q(x)$ over the interval $a \le x \le b$: (408)

$$Q(b) - Q(a) = \int_a^b Q'(x)\,dx$$

Area between two curves: (417)



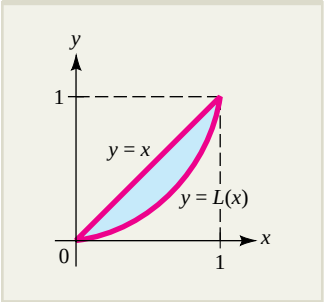

Area of $R$

$$= \int_a^b [f(x) - g(x)]\,dx$$

Average value of a function $f(x)$ over an interval $a \le x \le b$: (424)

$$V = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Lorentz curve (421, 422)




$$\text{Gini index} = 2\int_0^1 [x - L(x)]\,dx$$

Net excess profit (419)
Future value (amount) of an income stream (435)
Present value of an income stream (436)
Consumers' willingness to spend (437)
Consumers' surplus: (440)

$$\text{CS} = \int_0^{q_0} D(q)\,dq - p_0q_0, \text{ where } p = D(q) \text{ is demand}$$

Producers' surplus: (440)

$$\text{PS} = p_0q_0 - \int_0^{q_0} S(q)\,dq, \text{ where } p = S(q) \text{ is supply}$$

Survival and renewal (447)
Flow of blood through an artery (448)
Cardiac output (449)
Population from population density (452)
Volume of a solid revolution (455)

## Checkup for Chapter 5

**1.** Find these indefinite integrals (antiderivatives).

**a.** $\displaystyle\int x^3 - \sqrt{3x} + 5e^{-2x}\,dx$

**b.** $\displaystyle\int \frac{x^2 - 2x + 4}{x}\,dx$

**c.** $\displaystyle\int \sqrt{x}\left(x^2 - \frac{1}{x}\right)dx$

**d.** $\displaystyle\int \frac{x}{(3 + 2x^2)^{3/2}}\,dx$

**e.** $\displaystyle\int \frac{\ln\sqrt{x}}{x}\,dx$

**f.** $\displaystyle\int xe^{1+x^2}\,dx$

2. Evaluate each of these definite integrals.

   a. $\int_1^4 x^{3/2} + \frac{2}{x}\,dx$

   b. $\int_0^3 e^{3-x}\,dx$

   c. $\int_0^1 \frac{x}{x+1}\,dx$

   d. $\int_0^3 \frac{(x+3)}{\sqrt{x^2+6x+4}}\,dx$

3. In each case, find the area of the specified region.

   a. The region bounded by the curve $y = x + \sqrt{x}$, the $x$ axis, and the lines $x = 1$ and $x = 4$.

   b. The region bounded by the curve $y = x^2 - 3x$ and the line $y = x + 5$.

4. Find the average value of the function $f(x) = \frac{x-2}{x}$ over the interval $1 \le x \le 2$.

5. **NET CHANGE IN REVENUE** The marginal revenue of producing $q$ units of a certain commodity is $R'(q) = q(10 - q)$ hundred dollars per unit. How much additional revenue is generated as the level of production is increased from 4 to 9 units?

6. **BALANCE OF TRADE** The government of a certain country estimates that $t$ years from now, imports will be increasing at the rate $I'(t)$ and exports at the rate $E'(t)$, both in billions of dollars per year, where

$$I'(t) = 12.5e^{0.2t} \quad \text{and} \quad E'(t) = 1.7t + 3$$

The trade deficit is $D(t) = I(t) - E(t)$. By how much will the trade deficit for this country change over the next 5 years? Will it increase or decrease during this time period?

7. **CONSUMERS' SURPLUS** Suppose $q$ hundred units of a certain commodity are demanded by consumers when the price is $p = 25 - q^2$ dollars per unit. What is the consumers' surplus for the commodity when the level of production is $q_0 = 4$ (400 units)?

8. **AMOUNT OF AN INCOME STREAM** Money is transferred continuously into an account at the constant rate of \$5,000 per year. The account earns interest at the annual rate of 5% compounded continuously. How much will be in the account at the end of 3 years?

9. **POPULATION GROWTH** Demographers estimate that the fraction of people who will still be residing in a particular town $t$ years after they arrive is given by the function $f(t) = e^{-0.02t}$. If the current population is 50,000 and new townspeople arrive at the rate of 700 per year, what will be the population 20 years from now?

10. **AVERAGE DRUG CONCENTRATION** A patient is injected with a drug, and $t$ hours later, the concentration of the drug remaining in the patient's bloodstream is given by

$$C(t) = \frac{0.3t}{(t^2+16)^{1/2}} \quad \text{mg/cm}^3$$

What is the average concentration of the drug during the first 3 hours after the injection?

## Review Exercises

*In Exercises 1 through 20, find the indicated indefinite integral.*

1. $\int (x^3 + \sqrt{x} - 9)\,dx$

2. $\int \left(x^{2/3} - \frac{1}{x} + 5 + \sqrt{x}\right) dx$

3. $\int (x^4 - 5e^{-2x})\,dx$

4. $\int \left(2\sqrt[3]{s} + \frac{5}{s}\right) ds$

5. $\int \left(\frac{5x^3 - 3}{x}\right) dx$

6. $\int \left(\frac{3e^{-x} + 2e^{3x}}{e^{2x}}\right) dx$

7. $\int \left(t^5 - 3t^2 + \frac{1}{t^2}\right) dt$

8. $\int (x+1)(2x^2 + \sqrt{x})\,dx$

9. $\int \sqrt{3x+1}\,dx$

**10.** $\int (3x + 1)\sqrt{3x^2 + 2x + 5}\, dx$

**11.** $\int (x + 2)(x^2 + 4x + 2)^5\, dx$

**12.** $\int \frac{x + 2}{x^2 + 4x + 2}\, dx$

**13.** $\int \frac{3x + 6}{(2x^2 + 8x + 3)^2}\, dx$

**14.** $\int (t - 5)^{12}\, dt$

**15.** $\int v(v - 5)^{12}\, dv$

**16.** $\int \frac{\ln(3x)}{x}\, dx$

**17.** $\int 5xe^{-x^2}\, dx$

**18.** $\int \left(\frac{x}{x - 4}\right) dx$

**19.** $\int \left(\frac{\sqrt{\ln x}}{x}\right) dx$

**20.** $\int \left(\frac{e^x}{e^x + 5}\right) dx$

*In Exercises 21 through 30, evaluate the indicated definite integral.*

**21.** $\int_0^1 (5x^4 - 8x^3 + 1)\, dx$

**22.** $\int_1^4 (\sqrt{t} + t^{-3/2})\, dt$

**23.** $\int_0^1 (e^{2x} + 4\sqrt[3]{x})\, dx$

**24.** $\int_1^9 \frac{x^2 + \sqrt{x} - 5}{x}\, dx$

**25.** $\int_{-1}^2 30(5x - 2)^2\, dx$

**26.** $\int_{-1}^1 \frac{(3x + 6)}{(x^2 + 4x + 5)^2}\, dx$

**27.** $\int_0^1 2te^{t^2 - 1}\, dt$

**28.** $\int_0^1 e^{-x}(e^{-x} + 1)^{1/2}\, dx$

**29.** $\int_0^{e-1} \left(\frac{x}{x + 1}\right) dx$

**30.** $\int_e^{e^2} \frac{1}{x(\ln x)^2}\, dx$

**AREA BETWEEN CURVES** *In Exercises 31 through 38, sketch the indicated region R and find its area by integration.*

**31.** $R$ is the region under the curve $y = x + 2\sqrt{x}$ over the interval $1 \le x \le 4$.

**32.** $R$ is the region under the curve $y = e^x + e^{-x}$ over the interval $-1 \le x \le 1$.

**33.** $R$ is the region under the curve $y = \frac{1}{x} + x^2$ over the interval $1 \le x \le 2$.

**34.** $R$ is the region under the curve $y = \sqrt{9 - 5x^2}$ over the interval $0 \le x \le 1$.

**35.** $R$ is the region bounded by the curve $y = \frac{4}{x}$ and the line $x + y = 5$.

**36.** $R$ is the region bounded by the curves $y = \frac{8}{x}$ and $y = \sqrt{x}$ and the line $x = 8$.

**37.** $R$ is the region bounded by the curve $y = 2 + x - x^2$ and the $x$ axis.

**38.** $R$ is the triangular region with vertices (0, 0), (2, 4), and (0, 6).

**AVERAGE VALUE OF A FUNCTION** *In Exercises 39 through 42, find the average value of the given function over the indicated interval.*

**39.** $f(x) = x^3 - 3x + \sqrt{2x}$; over $1 \le x \le 8$

**40.** $f(t) = t\sqrt[3]{8 - 7t^2}$; over $0 \le t \le 1$

**41.** $g(v) = ve^{-v^2}$; over $0 \le v \le 2$

**42.** $h(x) = \frac{e^x}{1 + 2e^x}$; over $0 \le x \le 1$

**CONSUMERS' SURPLUS** *In Exercises 43 through 46, $p = D(q)$ is the demand curve for a particular commodity; that is, $q$ units of the commodity will be demanded when the price is $p = D(q)$ dollars per unit. In each case, for the given level of production $q_0$, find $p_0 = D(q_0)$ and compute the corresponding consumers' surplus.*

**43.** $D(q) = 4(36 - q^2)$; $q_0 = 2$ units

**44.** $D(q) = 100 - 4q - 3q^2$; $q_0 = 5$ units

**45.** $D(q) = 10e^{-0.1q}$; $q_0 = 4$ units

**46.** $D(q) = 5 + 3e^{-0.2q}$; $q_0 = 10$ units

**LORENTZ CURVES** *In Exercises 47 through 50, sketch the Lorentz curve $y = L(x)$ and find the corresponding Gini index.*

**47.** $L(x) = x^{3/2}$

**48.** $L(x) = x^{1.2}$

**49.** $L(x) = 0.3x^2 + 0.7x$

**50.** $L(x) = 0.75x^2 + 0.25x$

**SURVIVAL AND RENEWAL** *In Exercises 51 through 54, an initial population $P_0$ is given along with a renewal rate $R(t)$ and a survival function $S(t)$. In each case, use the given information to find the population at the end of the indicated term $T$.*

**51.** $P_0 = 75{,}000$; $R(t) = 60$; $S(t) = e^{-0.09t}$; $t$ in months; term $T = 6$ months

**52.** $P_0 = 125{,}000$; $R(t) = 250$; $S(t) = e^{-0.015t}$; $t$ in years; term $T = 5$ years

**53.** $P_0 = 100{,}000$; $R(t) = 90\,e^{0.1t}$; $S(t) = e^{-0.2t}$; $t$ in years; term $T = 10$ years

**54.** $P_0 = 200{,}000$; $R(t) = 50\,e^{0.12t}$; $S(t) = e^{-0.017t}$; $t$ in hours; term $T = 20$ hours

**VOLUME OF SOLID OF REVOLUTION** *In Exercises 55 through 58, find the volume of the solid of revolution formed by rotating the specified region $R$ about the $x$ axis.*

**55.** $R$ is the region under the curve $y = x^2 + 1$ from $x = -1$ to $x = 2$.

**56.** $R$ is the region under the curve $y = e^{-x/10}$ from $x = 0$ to $x = 10$.

**57.** $R$ is the region under the curve $y = \dfrac{1}{\sqrt{x}}$ from $x = 1$ to $x = 3$.

**58.** $R$ is the region under the curve $y = \dfrac{x+1}{\sqrt{x}}$ from $x = 1$ to $x = 4$.

*In Exercises 59 through 62, solve the given initial value problem.*

**59.** $\dfrac{dy}{dx} = 2$, where $y = 4$ when $x = -3$

**60.** $\dfrac{dy}{dx} = x(x - 1)$, where $y = 1$ when $x = 1$

**61.** $\dfrac{dx}{dt} = e^{-2t}$, where $x = 4$ when $t = 0$

**62.** $\dfrac{dy}{dt} = \dfrac{t+1}{t}$, where $y = 3$ when $t = 1$

**63.** Find the function whose tangent line has slope $x(x^2 + 1)^{-1}$ for each $x$ and whose graph passes through the point $(1, 5)$.

**64.** Find the function whose tangent line has slope $xe^{-2x^2}$ for each $x$ and whose graph passes through the point $(0, -3)$.

**65.** **NET ASSET VALUE** It is estimated that $t$ days from now a farmer's crop will be increasing at the rate of $0.5t^2 + 4(t + 1)^{-1}$ bushels per day. By how much will the value of the crop increase during the next 6 days if the market price remains fixed at \$2 per bushel?

**66.** **DEPRECIATION** The resale value of a certain industrial machine decreases at a rate that changes with time. When the machine is $t$ years old, the rate at which its value is changing is $200(t - 6)$ dollars per year. If the machine was bought new for \$12,000, how much will it be worth 10 years later?

**67.** **TICKET SALES** The promoters of a county fair estimate that $t$ hours after the gates open at 9:00 A.M. visitors will be entering the fair at the rate of $-4(t + 2)^3 + 54(t + 2)^2$ people per hour. How many people will enter the fair between 10:00 A.M. and noon?

**68.** **MARGINAL COST** At a certain factory, the marginal cost is $6(q - 5)^2$ dollars per unit when the level of production is $q$ units. By how much will the total manufacturing cost increase if the level of production is raised from 10 to 13 units?

**69.** **PUBLIC TRANSPORTATION** It is estimated that $x$ weeks from now, the number of commuters using a new subway line will be increasing at the rate of $18x^2 + 500$ per week. Currently, 8,000 commuters use the subway. How many will be using it 5 weeks from now?

**70.** **NET CHANGE IN BIOMASS** A protein with mass $m$ (grams) disintegrates into amino acids at a rate given by

$$\frac{dm}{dt} = \frac{-15t}{t^2 + 5}$$

What is the net change in mass of the protein during the first 4 hours?

**71. CONSUMPTION OF OIL** It is estimated that $t$ years from the beginning of the year 2005, the demand for oil in a certain country will be changing at the rate of $D'(t) = (1 + 2t)^{-1}$ billion barrels per year. Will more oil be consumed (demanded) during 2006 or during 2009? How much more?

**72. FUTURE VALUE OF AN INCOME STREAM** Money is transferred continuously into an account at the rate of $5{,}000e^{0.015t}$ dollars per year at time $t$ (years). The account earns interest at the annual rate of 5% compounded continuously. How much will be in the account at the end of 3 years?

**73. FUTURE VALUE OF AN INCOME STREAM** Money is transferred continuously into an account at the constant rate of \$1,200 per year. The account earns interest at the annual rate of 8% compounded continuously. How much will be in the account at the end of 5 years?

**74. PRESENT VALUE OF AN INCOME STREAM** What is the present value of an investment that will generate income continuously at a constant rate of \$1,000 per year for 10 years if the prevailing annual interest rate remains fixed at 7% compounded continuously?

**75. REAL ESTATE INVENTORY** In a certain community the fraction of the homes placed on the market that remain unsold for at least $t$ weeks is approximately $f(t) = e^{-0.2t}$. If 200 homes are currently on the market and if additional homes are placed on the market at the rate of 8 per week, approximately how many homes will be on the market 10 weeks from now?

**76. AVERAGE REVENUE** A bicycle manufacturer expects that $x$ months from now consumers will be buying 5,000 bicycles per month at the price of $P(x) = 200 + 3\sqrt{x}$ dollars per bicycle. What is the average revenue the manufacturer can expect from the sale of the bicycles over the next 16 months?

**77. NUCLEAR WASTE** A nuclear power plant produces radioactive waste at a constant rate of 300 pounds per year. The waste decays exponentially with a half-life of 35 years. How much of the radioactive waste from the plant will remain after 200 years?

**78. GROWTH OF A TREE** A tree has been transplanted and after $x$ years is growing at the rate of

$$h'(x) = 0.5 + \frac{1}{(x + 1)^2}$$

meters per year. By how much does the tree grow during the second year?

**79. FUTURE REVENUE** A certain oil well that yields 900 barrels of crude oil per month will run dry in 3 years. The price of crude oil is currently \$92 per barrel and is expected to rise at the constant rate of 80 cents per barrel per month. If the oil is sold as soon as it is extracted from the ground, what will be the total future revenue from the well?

**80. CONSUMERS' SURPLUS** Suppose that the consumers' demand function for a certain commodity is $D(q) = 50 - 3q - q^2$ dollars per unit.

**a.** Find the number of units that will be bought if the market price is \$32 per unit.

**b.** Compute the consumers' willingness to spend to get the number of units in part (a).

**c.** Compute the consumers' surplus when the market price is \$32 per unit.

**d.** Use the graphing utility of your calculator to graph the demand curve. Interpret the consumers' willingness to spend and the consumers' surplus as areas in relation to this curve.

**81. AVERAGE PRICE** Records indicate that $t$ months after the beginning of the year, the price of bacon in local supermarkets was $P(t) = 0.06t^2 - 0.2t + 6.2$ dollars per pound. What was the average price of bacon during the first 6 months of the year?

**82. SURFACE AREA OF A HUMAN BODY** The surface area $S$ of the body of an average person 4 feet tall who weighs $w$ lb changes at the rate

$$S'(w) = 110w^{-0.575} \quad \text{in}^2/\text{lb}$$

The body of a particular child who is 4 feet tall and weighs 50 lb has surface area 1,365 in$^2$. If the child gains 3 lb while remaining the same height, by how much will the surface area of the child's body increase?

**83. TEMPERATURE CHANGE** At $t$ hours past midnight, the temperature $T$ (°C) in a certain northern city is found to be changing at a rate given by

$$T'(t) = -0.02(t - 7)(t - 14) \quad \text{°C/hour}$$

By how much does the temperature change between 8 A.M. and 8 P.M.?

**84. EFFECT OF A TOXIN** A toxin is introduced to a bacterial colony, and $t$ hours later, the population $P(t)$ of the colony is changing at the rate

$$\frac{dP}{dt} = -(\ln 3)3^{4-t}$$

If there were 1 million bacteria in the colony when the toxin was introduced, what is $P(t)$? [*Hint:* Note that $3^x = e^{x \ln 3}$.]

**85. MARGINAL ANALYSIS** In a certain section of the country, the price of large Grade A eggs is currently \$2.50 per dozen. Studies indicate that $x$ weeks from now, the price $p(x)$ will be changing at the rate of $p'(x) = 0.2 + 0.003x^2$ cents per week.

**a.** Use integration to find $p(x)$ and then use the graphing utility of your calculator to sketch the graph of $p(x)$. How much will the eggs cost 10 weeks from now?

**b.** Suppose the rate of change of the price were $p'(x) = 0.3 + 0.003x^2$. How does this affect $p(x)$? Check your conjecture by sketching the new price function on the same screen as the original. Now how much will the eggs cost in 10 weeks?

**86. INVESTING IN A DOWN MARKET PERIOD** Jan opens a stock account with \$5,000 at the beginning of January and subsequently, deposits \$200 a month. Unfortunately, the market is depressed, and she finds that $t$ months after depositing a dollar, only $100f(t)$ cents remain, where $f(t) = e^{-0.01t}$. If this pattern continues, what will her account be worth after 2 years? [*Hint:* Think of this as a survival and renewal problem.]

**87. DISTANCE AND VELOCITY** After $t$ minutes, an object moving along a line has velocity $v(t) = 1 + 4t + 3t^2$ meters per minute. How far does the object travel during the third minute?

**88. AVERAGE POPULATION** The population (in thousands) of a certain city $t$ years after January 1, 1995, is given by the function

$$P(t) = \frac{150e^{0.03t}}{1 + e^{0.03t}}$$

What is the average population of the city during the decade 1995–2005?

**89. DISTRIBUTION OF INCOME** A study suggests that the distribution of incomes for social workers and physical therapists may be represented by the Lorentz curves $y = L_1(x)$ and $y = L_2(x)$, respectively, where

$$L_1(x) = x^{1.6} \quad \text{and} \quad L_2(x) = 0.65x^2 + 0.35x$$

For which profession is the distribution of income more equitable?

**90. DISTRIBUTION OF INCOME** A study conducted by a certain state determines that the Lorentz curves for high school teachers and real estate brokers are given by the functions

$$L_1(x) = 0.67x^4 + 0.33x^3$$
$$L_2(x) = 0.72x^2 + 0.28x$$

respectively. For which profession is the distribution of income more equitable?

**91. CONSERVATION** A lake has roughly the same shape as the bottom half of the solid formed by rotating the curve $2x^2 + 3y^2 = 6$ about the $x$ axis, for $x$ and $y$ measured in miles. Conservationists want the lake to contain 1,000 trout per cubic mile. If the lake currently contains 5,000 trout, how many more must be added to meet this requirement?

**92. HORTICULTURE** A sprinkler system sprays water onto a garden in such a way that $11e^{-r^2/10}$ inches of water per hour are delivered at a distance of $r$ feet from the sprinkler. What is the total amount of water laid down by the sprinkler within a 5-foot radius during a 20-minute watering period?

**93. SPEED AND DISTANCE** A car is driven so that after $t$ hours its speed is $S(t)$ miles per hour.

**a.** Write down a definite integral that gives the average speed of the car during the first $N$ hours.

**b.** Write down a definite integral that gives the total distance the car travels during the first $N$ hours.

**c.** Discuss the relationship between the integrals in parts (a) and (b).

**94.** Use the graphing utility of your calculator to draw the graphs of the curves $y = -x^3 - 2x^2 + 5x - 2$ and $y = x \ln x$ on the same screen. Use **ZOOM** and **TRACE** or some other feature of your calculator to find where the curves intersect, and then compute the area of the region bounded by the curves.

**95.** Repeat Exercise 94 for the curves

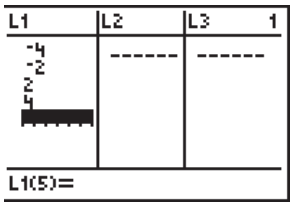
$$y = \frac{x-2}{x+1} \quad \text{and} \quad y = \sqrt{25 - x^2}$$

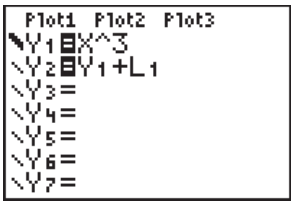
# EXPLORE! UPDATE

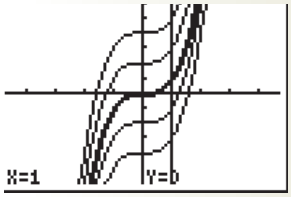
Complete solutions for all EXPLORE! boxes throughout the text can be accessed at the book-specific website, www.mhhe.com/hoffmann.

## Solution for Explore! on Page 373

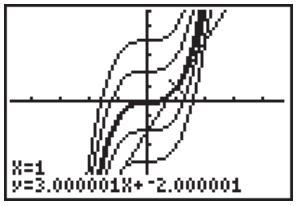
Store the constants $\{-4, -2, 2, 4\}$ into L1 and write Y1 = X^3 and Y2 = Y1 + L1. Graph Y1 in bold, using the modified decimal window $[-4.7, 4.7]1$ by $[-6, 6]1$. At $x = 1$ (where we have drawn a vertical line), the slopes for each curve appear equal.

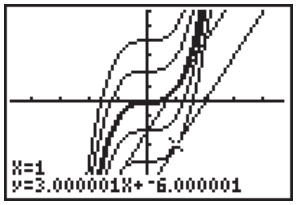


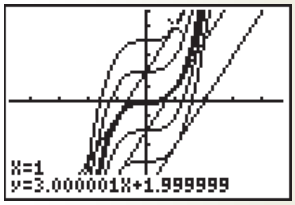




Using the tangent line feature of your graphing calculator, draw tangent lines at $x = 1$ for several of these curves. Every tangent line at $x = 1$ has a slope of 3, although each line has a different $y$ intercept.

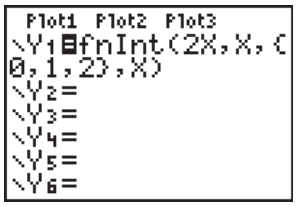


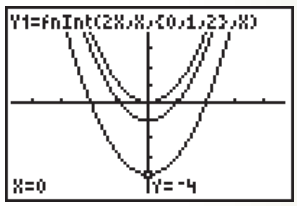




## Solution for Explore! on Page 374

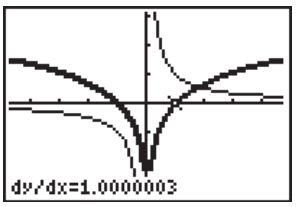
The numerical integral, **fnInt**(expression, variable, lower limit, upper limit) can be found via the **MATH** key, **9:fnInt(,** which we use to write Y1 below. We obtain a family of graphs that appear to be parabolas with vertices on the $y$ axis at $y = 0, -1$, and $-4$. The antiderivative of $f(x) = 2x$ is $F(x) = x^2 + C$, where $C = 0, -1$, and $-4$, in our case.

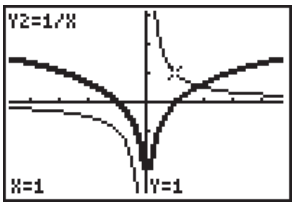


**Solution for Explore! on Page 375**

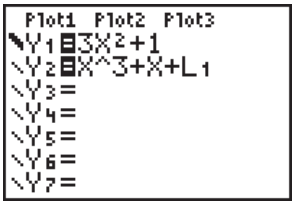
Place $y = F(x) = \ln |x|$ into Y1 as ln(abs($x$)), using a bold graphing style, and store $f(x) = \dfrac{1}{x}$ into Y2; then graph using a decimal window. Choose $x = 1$ and compare the derivative $F'(1)$, which is displayed in the graph on the left as $\dfrac{dy}{dx} = 1.0000003$, with the value $y = 1$ of $f(1)$ displayed on the right. The negligible difference in value in this case can be attributed to the use of numerical differentiation. In general, choosing any other nonzero $x$ value, we can verify that $F'(x) = f(x)$. For instance, when $x = -2$, we have $F'(-2) = -0.5 = f(-2)$.

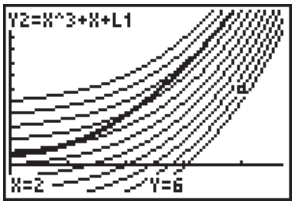


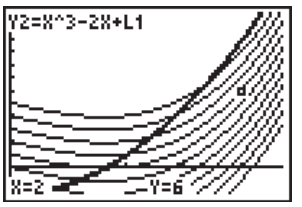




**Solution for Explore! on Page 377**

Place the integers from $-5$ to 5 into L1 (**STAT EDIT 1**). Set up the functions in the equation editor as shown here. Now graph with the designated window and notice that the antiderivative curves are generated sequentially from the lower to the upper levels. **TRACE** to the point (2, 6) and observe that the antiderivative that passes through this point is the second on the listing of L1. This curve is $F(x) = x^3 + x - 4$, which can also be calculated analytically, as in Example 5.1.4. For $f(x) = 3x^2 - 2$, the family of antiderivatives is $F(x) = x^3 - 2x + L1$ and the same window dimensions can be used to produce the screen on the right. The desired antiderivative is the eighth in L1, corresponding to $F(x) = x^3 - 2x + 2$, whose constant term can also be confirmed algebraically.

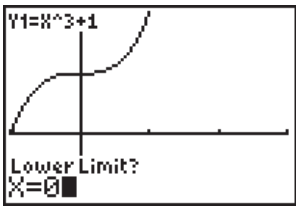


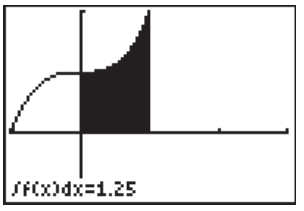



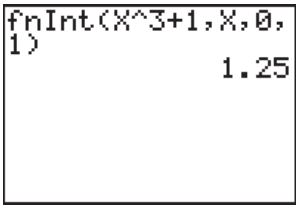




**Solution for Explore! on Page 403**

Following Example 5.3.3, set Y1 $= x^3 + 1$ and graph using a window $[-1, 3]1$ by $[-1, 2]1$. Access the numerical integration feature through **CALC, 7:∫f(x) dx**, specifying the lower limit as X = 0 and the upper limit as X = 1 to obtain $\int_0^1 (x^3 + 1)\, dx = 1.25$. Numerical integration can also be performed from the home screen via **MATH, 9: fnInt(,** as shown in the screen on the right.

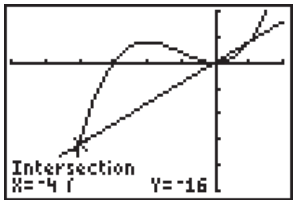


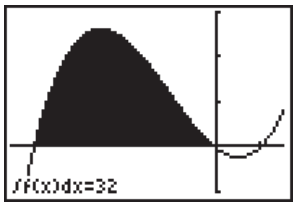



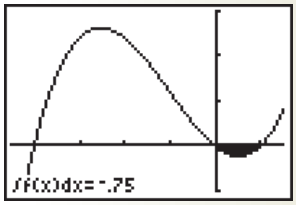



## Solution for Explore! on Page 418

Following Example 5.4.2, set Y1 = 4X and Y2 = $X^3 + 3X^2$ in the equation editor of your graphing calculator. Graph using the window [−6, 2]1 by [−25, 10]5. The points of intersection are at $x = -4$, 0, and 1. Considering $y = 4x$ as a horizontal baseline, the area between Y1 and Y2 can be viewed as that of the difference curve Y3 = Y2 − Y1. Deselect (turn off) Y1 and Y2 and graph Y3 using the window [−4.5, 1.5]0.5 by [−5, 15]5. Numerical integration applied to this curve between $x = -4$ and 0 yields an area of 32 square units for the first sector enclosed by the two curves. The area of the second sector, between $x = 0$ and 1, has area −0.75. The total area enclosed by the two curves is $32 + |-0.75| = 32.75$.

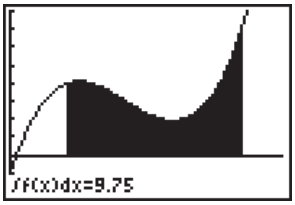


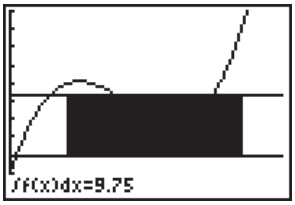


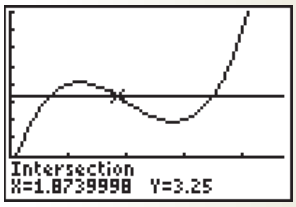



## Solution for Explore! on Page 424

Set Y1 = $x^3 - 6x^2 + 10x - 1$ and use the **CALC, 7:∫f(x) dx** feature to determine that the area under the curve from $x = 1$ to $x = 4$ is 9.75 square units, which equals the rectangular portion under Y2 = 9.75/(4 − 1) = 3.25 of length 3. It is as though the area under $f(x)$ over [1, 4] turned to water and became a level surface of height 3.25, the average $f(x)$ value. This value is attained at $x \approx 1.874$ (shown on the right) and also at $x = 3.473$. Note that you must clear the previous shading, using **DRAW, 1:ClrDraw,** before constructing the next drawing.

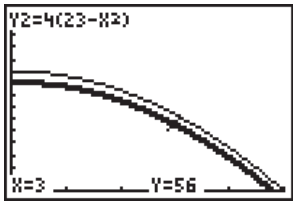


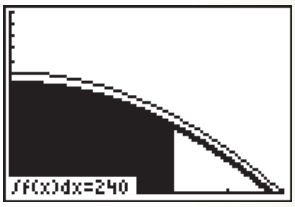





## Solution for Explore! on Page 438

We graph $D_{\text{new}}(q)$ in bold as Y2 = $4(23 - X^2)$ with $D(q)$ in Y1 = $4(25 - X^2)$, using the viewing window [0, 5]1 by [0, 150]10. Visually, $D_{\text{new}}(q)$ is less than $D(q)$ for the observable range of values, supporting the conjecture that the area under the curve of $D_{\text{new}}(q)$ will be less than that of $D(q)$ over the range of values [0, 3]. This area is calculated to be \$240, less than the \$264 shown for $D(q)$ in Figure 5.20.





# THINK ABOUT IT

## JUST NOTICEABLE DIFFERENCES IN PERCEPTION

Calculus can help us answer questions about human perception, including questions relating to the number of different frequencies of sound or the number of different hues of light people can distinguish (see the accompanying figure). Our present goal is to show how integral calculus can be used to estimate the number of steps a person can distinguish as the frequency of sound increases from the lowest audible frequency of 15 hertz (Hz) to the highest audible frequency of 18,000 Hz. (Here hertz, abbreviated Hz, equals cycles per second.)

A mathematical model* for human auditory perception uses the formula $y = 0.767x^{0.439}$, where $y$ Hz is the smallest change in frequency that is detectable at frequency $x$ Hz. Thus, at the low end of the range of human hearing, 15 Hz, the smallest change of frequency a person can detect is $y = 0.767 \times 15^{0.439} \approx 2.5$ Hz, while at the upper end of human hearing, near 18,000 Hz, the least noticeable difference is approximately $y = 0.767 \times 18{,}000^{0.439} \approx 57$ Hz. If the smallest noticeable change of frequency were the same for all frequencies that people can hear, we could find the number of noticeable steps in human hearing by simply dividing the total frequency range by the size of this smallest noticeable change. Unfortunately, we have just seen that the smallest noticeable change of frequency increases as frequency increases, so the simple approach will not work. However, we can estimate the number of distinguishable steps using integration.

Toward this end, let $y = f(x)$ represent the just noticeable difference of frequency people can distinguish at frequency $x$. Next, choose numbers $x_0, x_1, \ldots, x_n$ beginning at $x_0 = 15$ Hz and working up through higher frequencies to $x_n = 18{,}000$ Hz in such a way that for $j = 0, 2, \ldots, n - 1$,

$$x_j + f(x_j) = x_{j+1}$$

*Part of this essay is based on *Applications of Calculus: Selected Topics from the Environmental and Life Sciences*, by Anthony Barcellos, New York: McGraw-Hill, 1994, pp. 21–24.

In other words, $x_{j+1}$ is the number we get by adding the just noticeable difference at $x_j$ to $x_j$ itself. Thus, the $j$th step has length

$$\Delta x_j = x_{j+1} - x_j = f(x_j)$$

Dividing by $f(x_j)$, we get

$$\frac{\Delta x_j}{f(x_j)} = \frac{x_{j+1} - x_j}{f(x_j)} = 1$$

and it follows that

$$\begin{aligned}\sum_{j=0}^{n-1} \frac{\Delta x_j}{f(x_j)} &= \sum_{j=0}^{n-1} \frac{x_{j+1} - x_j}{f(x_j)} = \frac{x_1 - x_0}{f(x_0)} + \frac{x_2 - x_1}{f(x_1)} + \cdots + \frac{x_n - x_{n-1}}{f(x_n)} \\ &= \underbrace{1 + 1 + \cdots + 1}_{n \text{ terms}} = n\end{aligned}$$

The sum on the left side of this equation is a Riemann sum, and since the step sizes $\Delta x_j = x_{j+1} - x_j$ are very small, the sum is approximately equal to a definite integral. Specifically, we have

$$\int_{x_0}^{x_n} \frac{dx}{f(x)} \approx \sum_{j=0}^{n-1} \frac{\Delta x_j}{f(x_j)} = n$$

Finally, using the modeling formula $f(x) = 0.767x^{0.439}$ along with $x_0 = 15$ and $x_n = 18{,}000$, we find that

$$\begin{aligned}\int_{x_0}^{x_n} \frac{dx}{f(x)} &= \int_{15}^{18{,}000} \frac{dx}{0.767x^{0.439}} \\ &= \frac{1}{0.767}\left(\frac{x^{0.561}}{0.561}\right)\Bigg|_{15}^{18{,}000} \\ &= 2.324(18{,}000^{0.561} - 15^{0.561}) \\ &= 556.2\end{aligned}$$

Thus, there are approximately 556 just noticeable steps in the audible range from 15 Hz to 18,000 Hz.

Here are some questions in which you are asked to apply these principles to issues involving both auditory and visual perception.

## Questions

1. The 88 keys of a piano range from 15 Hz to 4,186 Hz. If the number of keys were based on the number of just noticeable differences, how many keys would a piano have?
2. An 8-bit gray-scale monitor can display 256 shades of gray. Let $x$ represent the darkness of a shade of gray, where $x = 0$ for white and $x = 1$ for totally black. One model for gray-scale perception uses the formula $y = Ax^{0.3}$, where $A$ is a positive constant and $y$ is the smallest change detectable by the human eye at gray-level $x$. Experiments show that the human eye is incapable of distinguishing as many as 256 different shades of gray, so the number $n$ of just noticeable shading differences from $x = 0$ to $x = 1$ must be less than 256. Using the

assumption that $n < 256$, find a lower bound for the constant $A$ in the modeling formula $y = Ax^{0.3}$.

**3.** One model of the ability of human vision to distinguish colors of different hue uses the formula $y = 2.9 \times 10^{-24}\, x^{8.52}$, where $y$ is the just noticeable difference for a color of wavelength $x$, with both $x$ and $y$ measured in nanometers (nm).

**a.** Blue-green light has a wavelength of 580 nm. What is the least noticeable difference at this wavelength?

**b.** Red light has a wavelength of 760 nm. What is the least noticeable difference at this wavelength?

**c.** How many just noticeable steps are there in hue from blue-green light to red light?

**4.** Find a model of the form $y = ax^k$ for just noticeable differences in hue for the color spectrum from blue-green light at 580 nm to violet light at 400 nm. Use the fact that the minimum noticeable difference at the wavelength of blue-green light is 1 nm, while at the wavelength of violet light, the minimum noticeable difference is 0.043 nm.