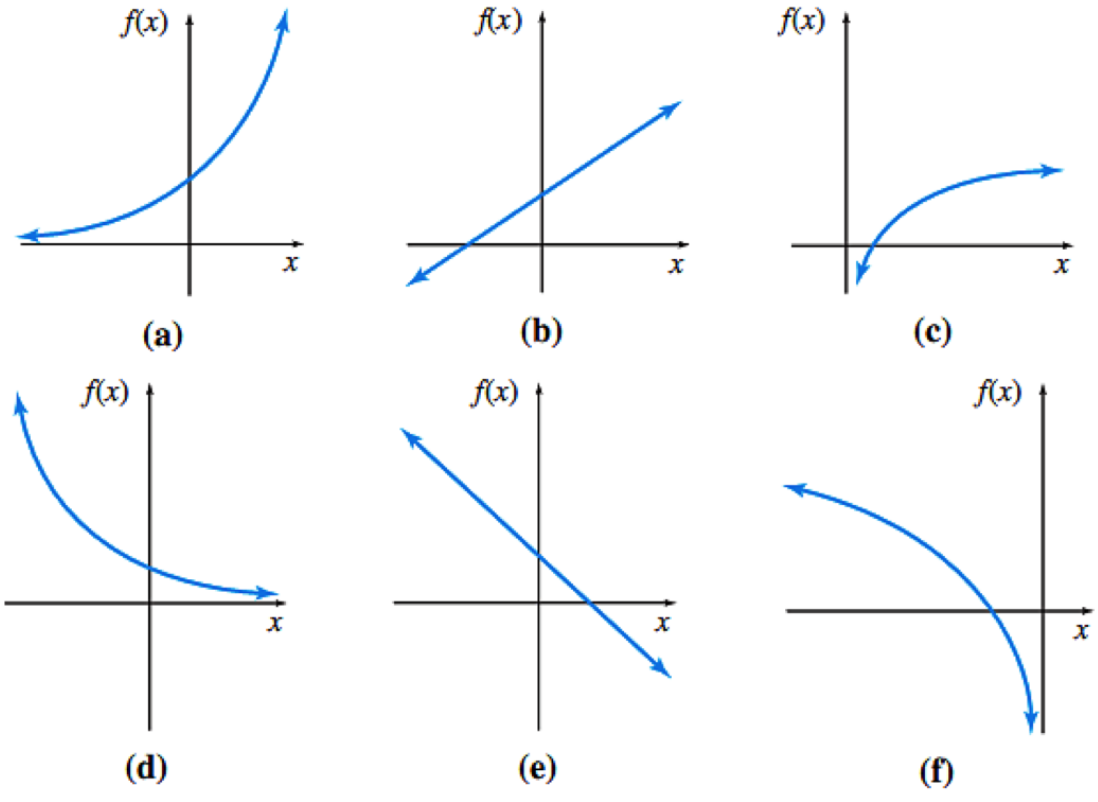
Increasing and Decreasing Functions

Let f be a function defined on some interval. Then for any two numbers x_1 and x_2 in the interval, f is **increasing** on the interval if

$$f(x_1) < f(x_2) \quad \text{whenever} \quad x_1 < x_2,$$

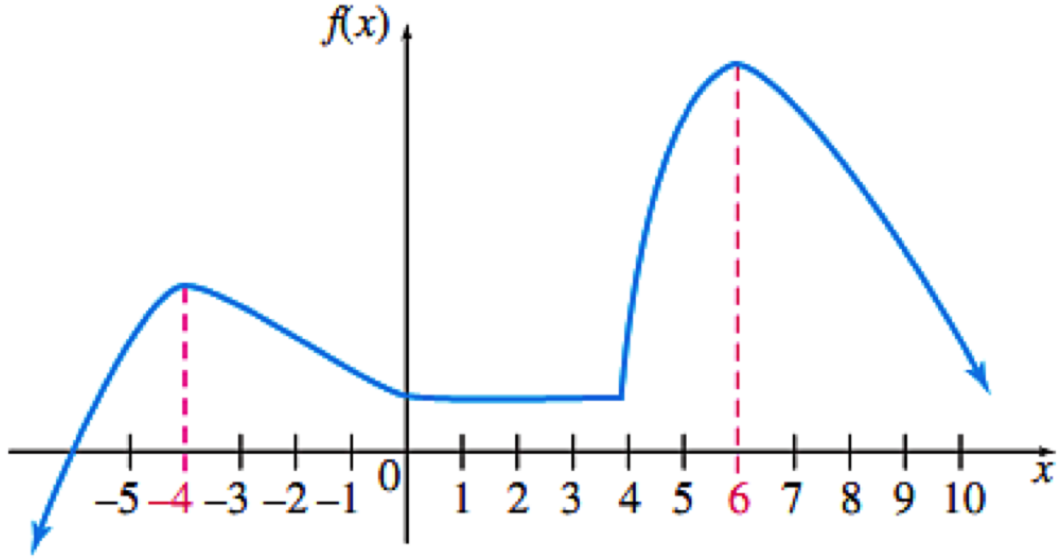
and f is **decreasing** on the interval if

$$f(x_1) > f(x_2) \quad \text{whenever} \quad x_1 < x_2.$$



Example

Find the intervals where the function is increasing and/or decreasing.



The graph is increasing on the interval $(-\infty, -4)$.

The graph is decreasing on the interval $(-4, 0)$.

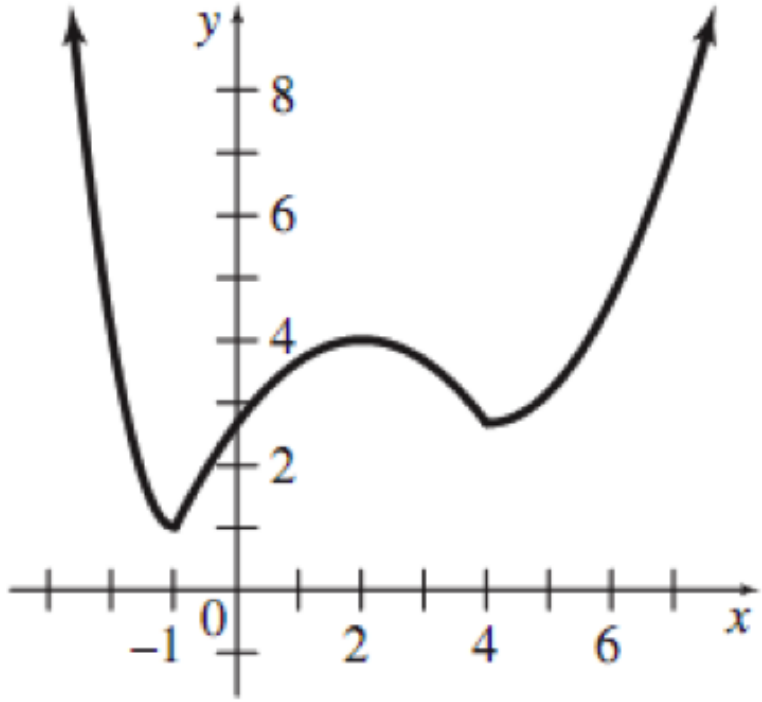
The graph is constant on the interval $(0, 4)$.

The graph is increasing on the interval $(4, 6)$.

The graph is decreasing on the interval $(6, \infty)$.

Example

Find the intervals where the function is increasing and/or decreasing.



The graph is increasing on the intervals $(-1, 2)$ and $(4, \infty)$.

The graph is decreasing on the intervals $(-\infty, -1)$ and $(2, 4)$.

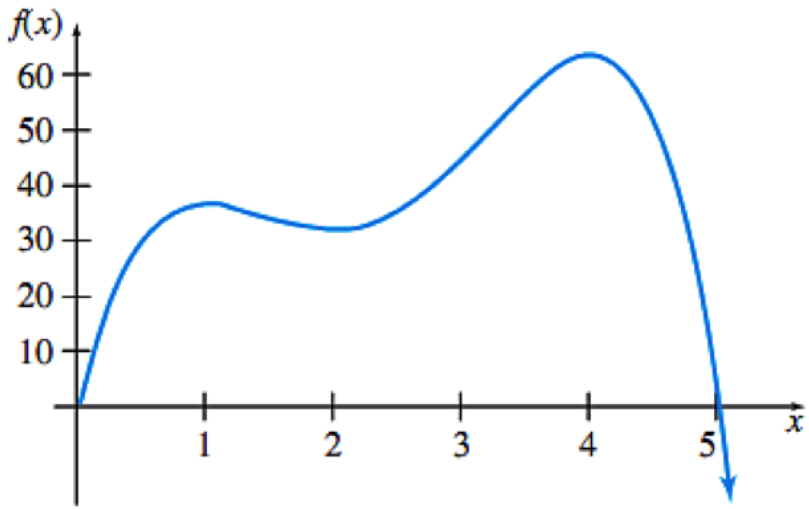
Test for Intervals Where $f(x)$ is Increasing and Decreasing

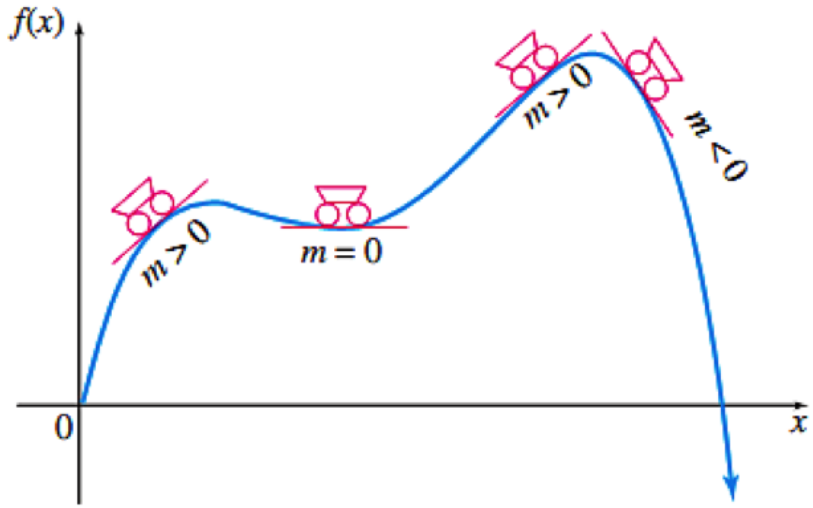
Suppose a function f has a derivative at each point in an open interval; then

if $f'(x) > 0$ for each x in the interval, f is *increasing* on the interval; ↗

if $f'(x) < 0$ for each x in the interval, f is *decreasing* on the interval; ↘

if $f'(x) = 0$ for each x in the interval, f is *constant* on the interval. →





Critical Numbers

The **critical numbers** for a function f are those numbers c in the domain of f for which $f'(c) = 0$ or $f'(c)$ does not exist. A **critical point** is a point whose x-coordinate is the critical number c and whose y-coordinate is $f(c)$.

Applying the Test

1. Locate the critical numbers for f on a number line, as well as any points where f is undefined. These points determine several open intervals.
2. Choose a value of x in each of the intervals determined in Step 1. Use these values to decide whether $f'(x) > 0$ or $f'(x) < 0$ in that interval.
3. Use the test on the previous page to decide whether f is increasing or decreasing on the interval.

As you can see, this process is simply a sign analysis.

Example

Find the intervals in which $f(x) = x^3 + 3x^2 - 9x + 4$ is increasing or decreasing. Graph the function.

To find the critical numbers, first find the derivative.

$$f'(x) = 3x^2 + 6x - 9$$

Set the derivative equal to 0 and solve the resulting equation by factoring.

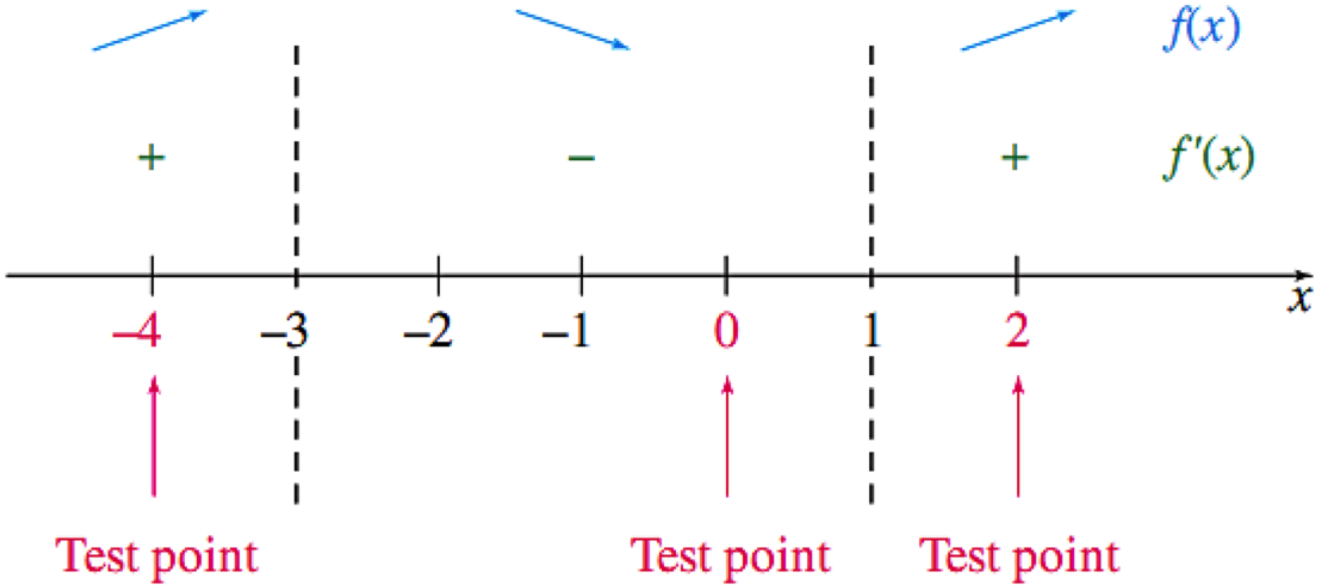
$$\begin{aligned} 3x^2 + 6x - 9 &= 0 \\ 3(x^2 + 2x - 3) &= 0 \\ 3(x-1)(x+3) &= 0 \\ x = 1 \quad &or \quad x = -3 \end{aligned}$$

Don't forget that the definition of critical number is twofold. We also need to check for numbers in the domain of f where the derivative is undefined.

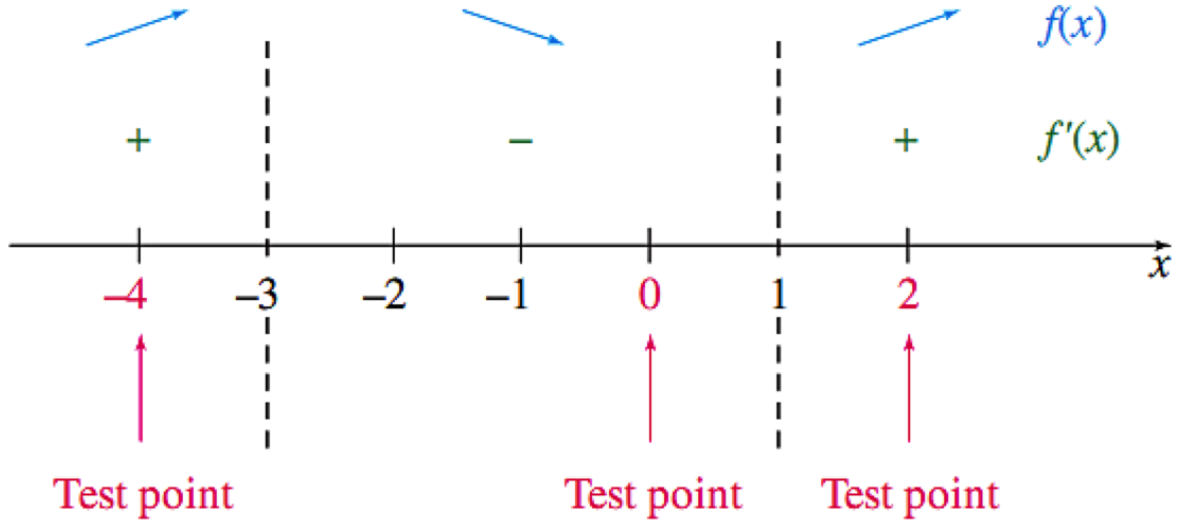
Since both f and f' are polynomials, they are never undefined. Thus there are exactly 2 critical numbers, $x = -3$ and $x = 1$.

To determine where the function is increasing or decreasing, make a number line containing the critical points, $x = -3$ and $x = 1$, and perform a sign analysis.

The number line should be broken up into three intervals, $(-\infty, -3)$, $(-3, 1)$, and $(1, \infty)$.



We can choose any values we want for x in the intervals, so for simplicity we can choose $x = -4, 0, 2$ as our test points.



$$f'(x) = 3(x-1)(x+3)$$

$$\begin{aligned} f'(-4) &= 3(-5)(-1) &&= +15 \\ f'(0) &= 3(-1)(3) &&= -9 \\ f'(2) &= 3(1)(5) &&= +15 \end{aligned}$$

Therefore the function $f(x) = x^3 + 3x^2 - 9x + 4$ is

increasing on the intervals $(-\infty, -3), (1, \infty)$ and

decreasing on the interval $(-3, 1)$.

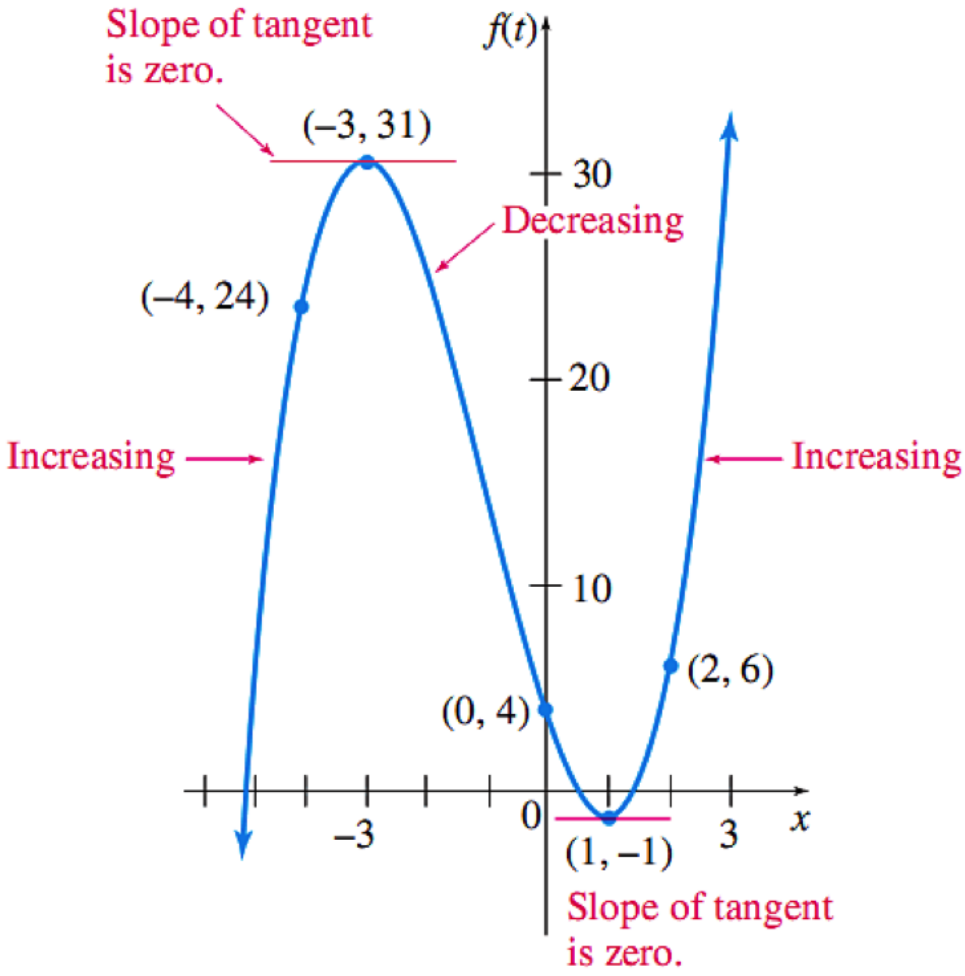
To graph the function, first compute the y-values for any "interesting" x's found earlier and plot those points.

$$f(x) = x^3 + 3x^2 - 9x + 4$$

$$\begin{aligned} f(-3) &= -27 + 27 + 27 + 4 = 31 \\ f(1) &= 1 + 3 - 9 + 4 = -1 \end{aligned}$$

$$\begin{aligned} f(-4) &= -64 + 48 + 36 + 4 = 24 \\ f(0) &= 0 + 0 - 0 + 4 = 4 \\ f(2) &= 8 + 12 - 18 + 4 = 6 \end{aligned}$$

Then use the inc/dec results found earlier to draw in the graph.


Slope of tangent
is zero.
f(t)
(–3, 31)
30
Decreasing
(–4, 24)
20
Increasing
Increasing
10
(2, 6)
(0, 4)
0
–3
3
x
(1, –1)
Slope of tangent
is zero.

$f(x) = x^3 + 3x^2 - 9x + 4$

Example

Find the intervals in which $f(x) = -x^3 - 2x^2 + 15x + 10$ is increasing or decreasing. Graph the function.

$$\begin{aligned} f'(x) &= -3x^2 - 4x + 15 \\ 0 &= -3x^2 - 4x + 15 \\ 0 &= -(3x - 5)(x + 3) \\ &\Rightarrow \quad 3x - 5 = 0 \quad or \quad x + 3 = 0 \\ &\Rightarrow \quad x = \frac{5}{3} \quad or \quad x = -3 \end{aligned}$$

Don't forget that the definition of critical number is twofold. We also need to check for numbers in the domain of f where the derivative is undefined.

Since both f and f' are polynomials, they are never undefined.

To determine where the function is increasing or decreasing, make a number line containing the critical points, $x = -3$ and $x = \frac{5}{3}$, and perform a sign analysis.

The number line should be broken up into three intervals, $(-\infty, -3)$, $(-3, \frac{5}{3})$, and $(\frac{5}{3}, \infty)$.

We can choose any values we want for x in the intervals, so for simplicity we can choose $x = -4, 0, 2$ as our test points.

$$f'(x) = -(3x-5)(x+3)$$

$$\begin{aligned} f'(-4) &= -(-)(-) &&= && - \\ f'(0) &= -(-)(+) &&= && + \\ f'(2) &= -(+)(+) &&= && - \end{aligned}$$

Therefore the function $f(x) = -x^3 - 2x^2 + 15x + 10$ is

decreasing on the intervals $(-\infty, -3)$,$(\frac{5}{3}, \infty)$ and

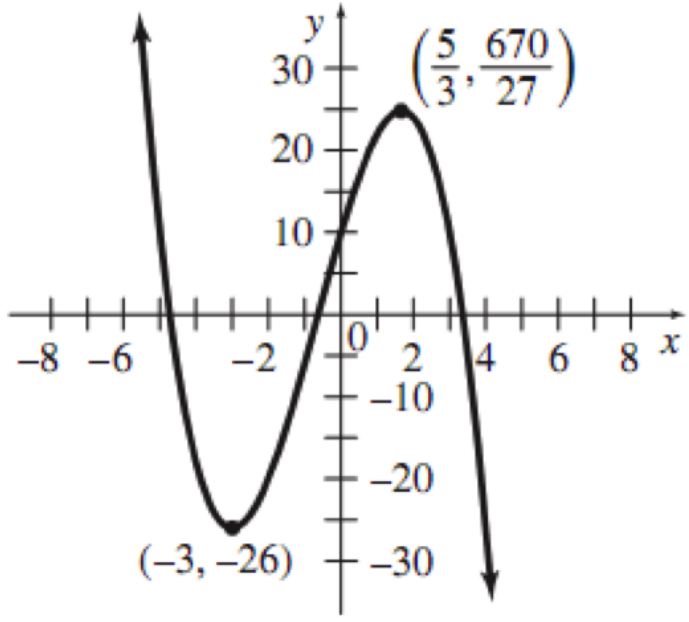
increasing on the interval $(-3, \frac{5}{3})$.

To graph $f(x) = -x^3 - 2x^2 + 15x + 10$, compute and plot:

$$\begin{aligned} f(-3) &= -26 \\ f(\tfrac{5}{3}) &= \frac{670}{27} \end{aligned}$$

and use the information found earlier;

Increasing on $(-3, 5/3)$. Decreasing on $(-\infty, -3)$ and $(5/3, \infty)$



Warning # 1 Just because you've found a critical point **does not** mean that the function will change from increasing to decreasing or vice versa.

Example

$$f(x) = x^3$$

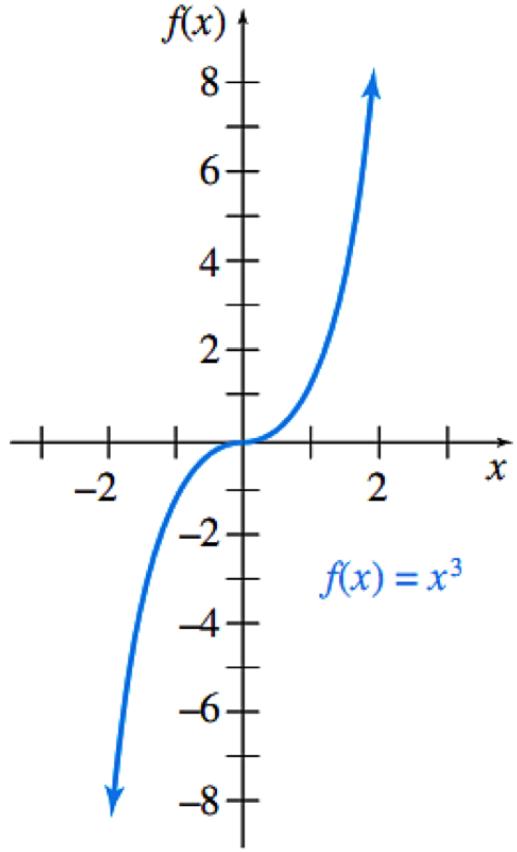
$f'(x) = 3x^2$ so there is exactly one critical number at $x = 0$.

Notice that the derivative is **NEVER** negative, so there is no change at the critical point.

Example

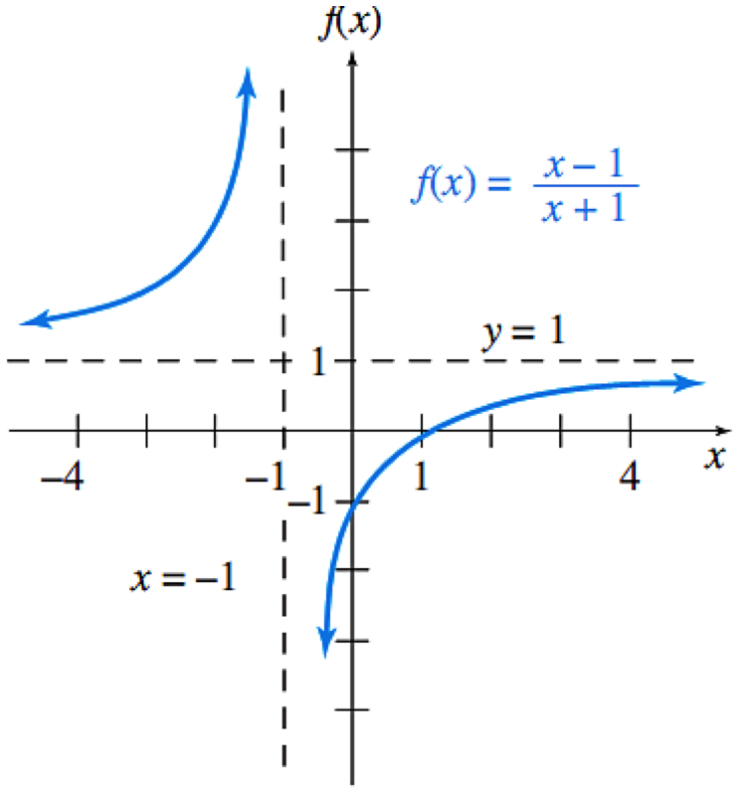
$$f(x) = x^3$$

This coincides with what we already know about the function, that it is **NEVER** decreasing.



Warning # 2 Just because a function doesn't have any critical points **does not** mean that you can not determine where the function is increasing/decreasing.

Example



$$f'(x) = \frac{x+1-x+1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

Example

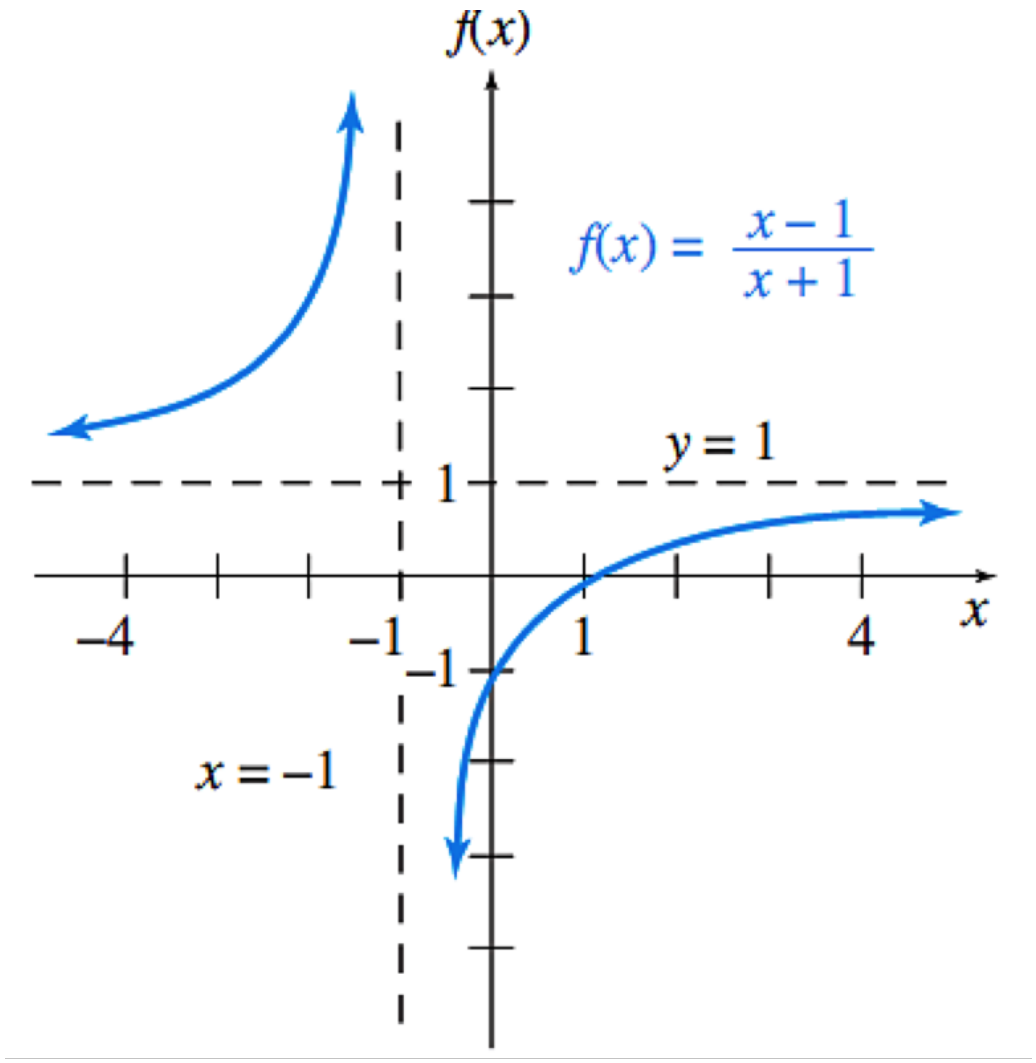
$$f(x) = \frac{x-1}{x+1}$$

$$f'(x) \quad = \quad \frac{2}{(x+1)^2} \quad = \quad 0 \quad ????$$

The derivative is undefined at $x = -1$, however that value is not in the domain of f.

Therefore there are no critical numbers.

Even though we did not find any critical numbers, we see that the derivative is always positive. Therefore the graph is always increasing.



The function is increasing on the intervals $(-\infty, -1), (-1, \infty)$.

Relative Maximum or Minimum

Let c be a number in the domain of a function f. Then $f(c)$ is a **relative** (or **local**) **maximum** for f if there exists an open interval (a, b) containing c such that

$$f(x) \leq f(c)$$

for all x in (a, b).

Likewise, $f(c)$ is a **relative** (or **local**) **minimum** for f if there exists an open interval (a, b) containing c such that

$$f(x) \geq f(c)$$

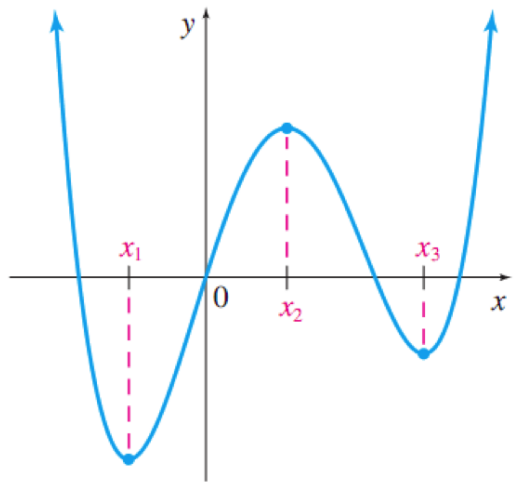
for all x in (a, b). (*continued*)

A function has a **relative** (or **local**) **extremum** (plural: **extrema**) at c if it has either a relative maximum or a relative minimum there.

If c is an endpoint of the domain of f, we only consider x in the half-open interval that is in the domain.*

Example

Identify the x-values of all points where the graph has relative extrema.



At $x = x_1$, there is a relative minimum because the graph changes from decreasing to increasing at x_1.

At $x = x_2$, there is a rel max because the graph changes from increasing to decreasing at x_2.

At $x = x_3$, there is a rel min because the graph changes from decreasing to increasing at x_3.

First Derivative Test

Let c be a critical number for a function f. Suppose that f is continuous on (a, b) and differentiable on (a, b) except possibly at c, and that c is the only critical number for f in (a, b).

1. $f(c)$ is a relative maximum of f if the derivative $f'(x)$ is positive in the interval (a, c) and negative in the interval (c, b).
2. $f(c)$ is a relative minimum of f if the derivative $f'(x)$ is negative in the interval (a, c) and positive in the interval (c, b).

This just means that there is a local maximum if the graph changes from increasing to decreasing.

And a local minimum if the graph changes from decreasing to increasing.

First Derivative Test

Let c be a critical number for a function f. Suppose that f is continuous on (a, b) and differentiable on (a, b) except possibly at c, and that c is the only critical number for f in (a, b).

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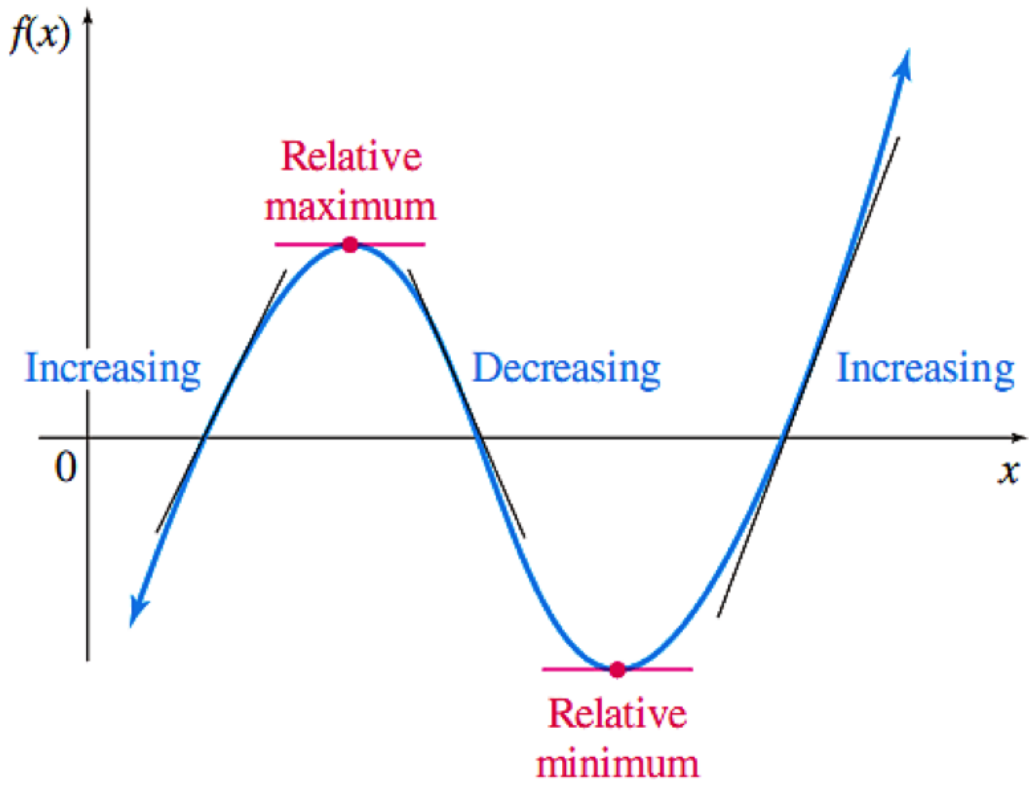
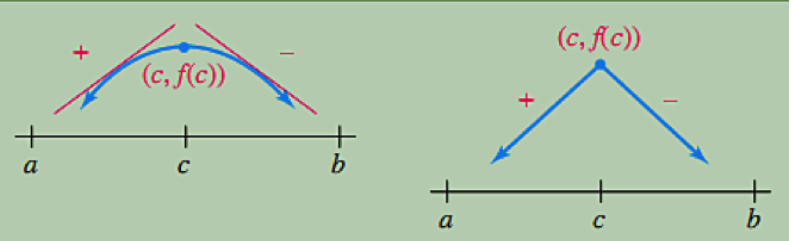
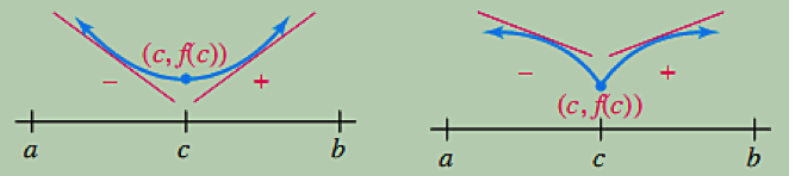
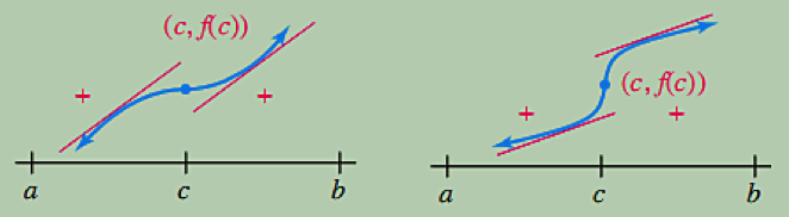
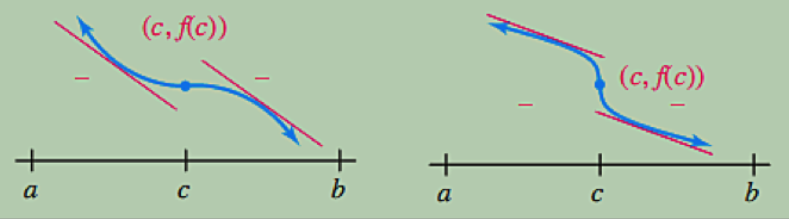


Relative Extrema

$f(x)$ has:	Sign of f' in (a, c)	Sign of f' in (c, b)	Sketches
Relative maximum	+	−	
Relative minimum	−	+	
No relative extrema	+	+	
No relative extrema	−	−	

Example

Find all relative extrema for $f(x) = x^3 + 3x^2 - 9x + 4$.

To find the critical numbers, first find the derivative.

$$f'(x) = 3x^2 + 6x - 9$$

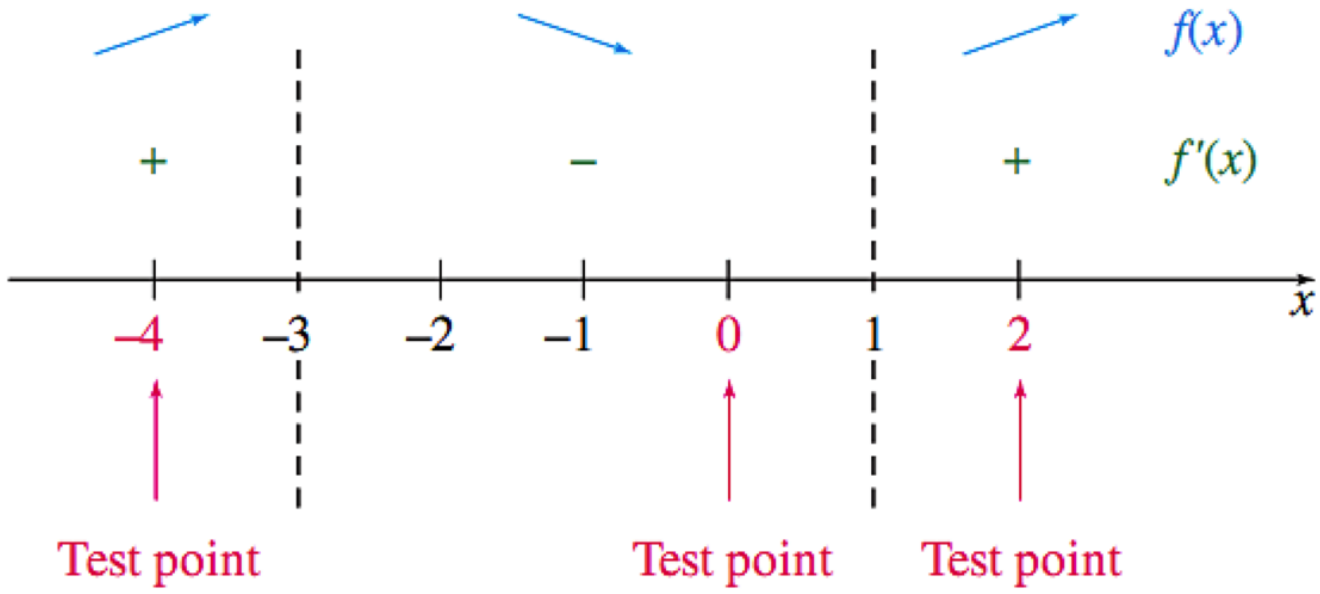
Set the derivative equal to 0 and solve the resulting equation by factoring.

$$\begin{aligned} 3x^2 + 6x - 9 &= 0 \\ 3(x^2 + 2x - 3) &= 0 \\ 3(x-1)(x+3) &= 0 \\ x = 1 \quad &or \quad x = -3 \end{aligned}$$

Don't forget that the definition of critical number is twofold.

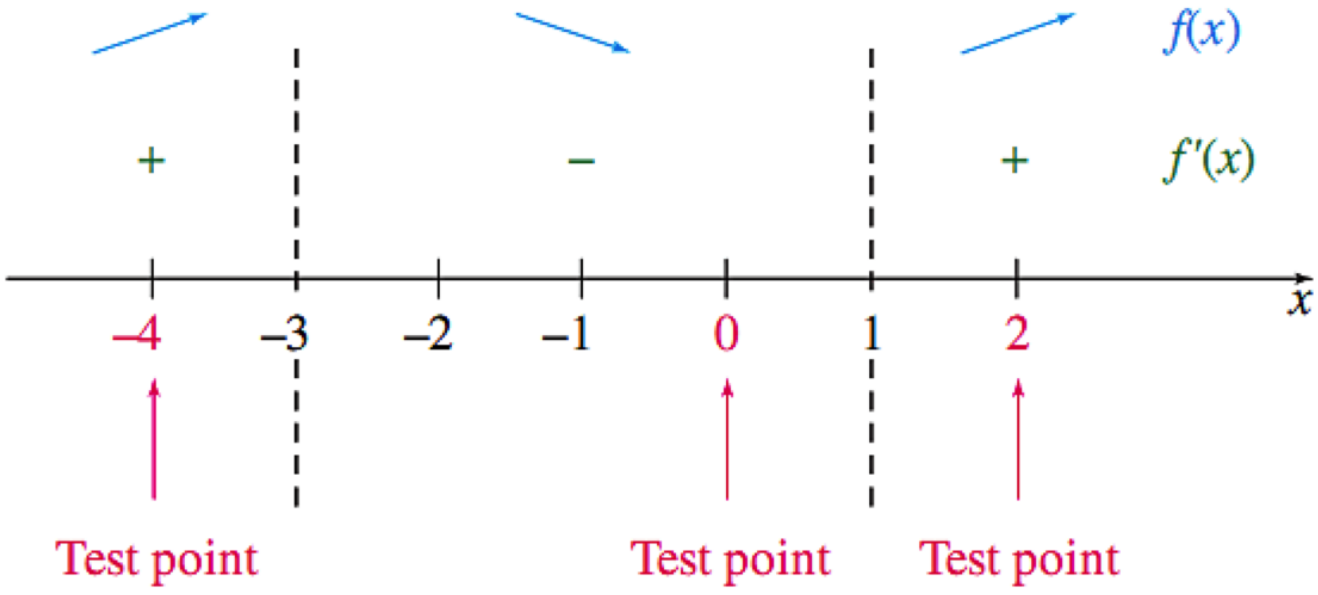
To determine where the function is increasing or decreasing, make a number line containing the critical points, $x = -3$ and $x = 1$, and perform a sign analysis.

We can choose $x = -4, 0, 2$ as our test points.



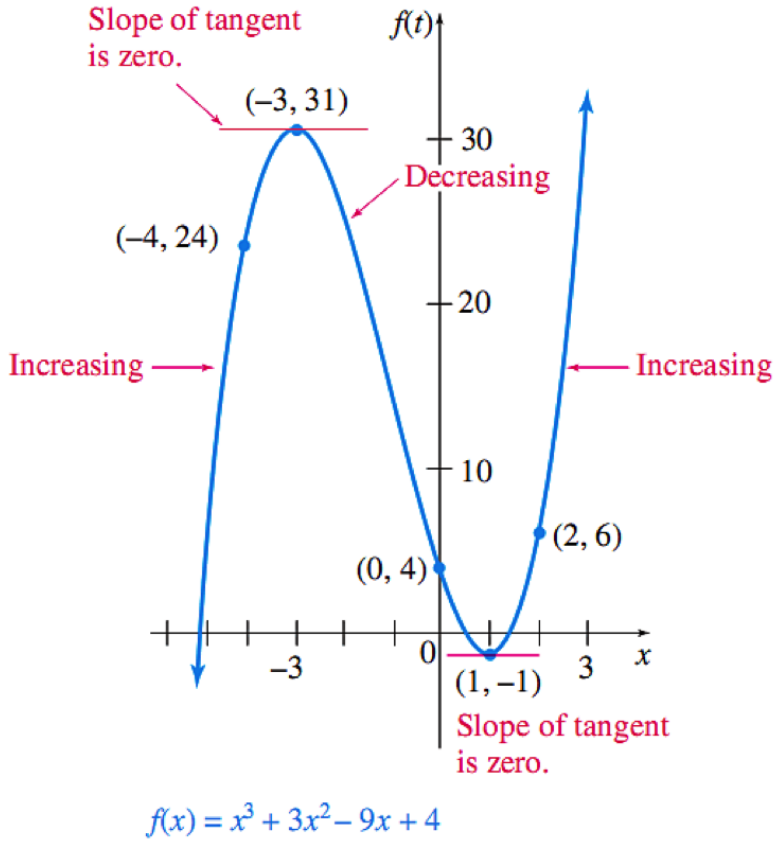
Since the derivative changes from positive to negative at $x = -3$, there is a relative maximum at $x = -3$.

Since the derivative changes from negative to positive at $x = 1$, there is a relative minimum at $x = 1$.



The value of the local maximum is $f(-3) = 31$.

The value of the local minimum is $f(1) = -1$.



The rel max occurs at the point $(-3, 31)$.

The rel min occurs at the point $(1, -1)$.

Example

Find all rel extrema for $f(x) = -x^3 - 2x^2 + 15x + 10.$

$$\begin{aligned} f'(x) &= -3x^2 - 4x + 15 \\ 0 &= -3x^2 - 4x + 15 \\ 0 &= -(3x - 5)(x + 3) \\ &\Rightarrow \quad 3x - 5 = 0 \quad or \quad x + 3 = 0 \\ &\Rightarrow \quad x = \frac{5}{3} \quad or \quad x = -3 \end{aligned}$$

Don't forget that the definition of critical number is twofold.

To determine where the function is increasing or decreasing, make a number line containing the critical points, $x = -3$ and $x = \frac{5}{3}$, and perform a sign analysis.

The function $f(x) = -x^3 - 2x^2 + 15x + 10$ is

Decreasing on the interval $(-\infty, -3)$.

Increasing on the interval $(-3, \frac{5}{3})$.

Decreasing on the interval $(\frac{5}{3}, \infty)$.

There is a local min at $x = -3$ and a local max at $x = \frac{5}{3}$.