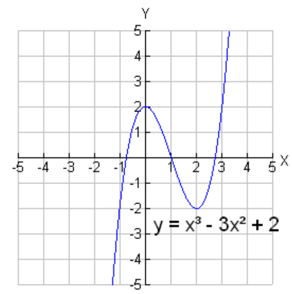
THE CALCULUS OF CURVE SKETCHING

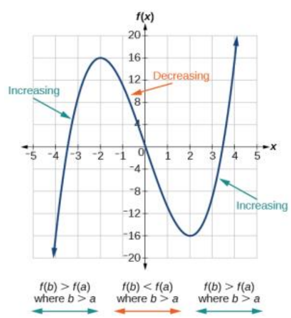
Here is a summary of the analytical methods of calculus to sketch graphs and interpret them. It starts with polynomial functions, which are *differentiable* at all values of their domain (the real numbers).

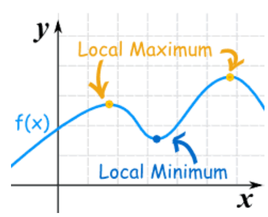


Because of this "good behavior" of polynomials, they are ideal for investigating critical numbers of a function, intervals of increase and decrease, extremes, concavity and points of inflection.

Definitions

- c is a *critical number* of function $f(x)$ if either $f'(c) = 0$ or $f'(c)$ does not exist (DNE).
- A function may have a *local extreme* at a critical number c (including a non-differentiable cusp or corner), or it may have an *inflection point*.
- A function $f(x)$ is (strictly) *increasing* on an interval I if for each a, b in I, when $a < b$, $f(a) < f(b)$.
- A function $f(x)$ is (strictly) *decreasing* on an interval I if for each a, b in I, when $a < b$, $f(a) > f(b)$.



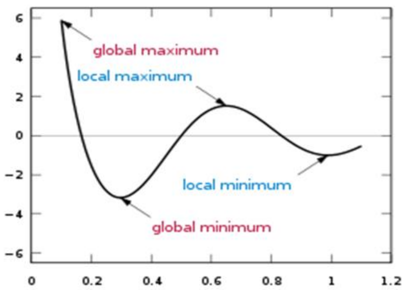


On an interval where $f(x)$ is increasing, $f'(x) > 0$; on an interval where $f(x)$ is decreasing, $f'(x) < 0$.

- $f(x)$ has a *local (relative) maximum* at a if $f(x) \leq f(a)$ for all x in an arbitrarily small interval (called an ε-neighborhood) of a.
- $f(x)$ has a *local (relative) minimum* at a if $f(x) \geq f(a)$ for all x in an an ε-neighborhood of a.

[*Note*: A *constant* function on I has both a local a max and a local min at every x on that interval.]

- Local maximum and minimum values of a function are called *local extremes* of the function.
- M is a *global (absolute) maximum* if for every x in I, $f(x) \leq M$; m is a *global (absolute) minimum* if for every x in I, $f(x) \geq m$.



First derivative test (FDT): Is $f(c)$ a local max or min?

Visually, for some critical number c, $f(c)$ is a local max if f is increasing when $x < c$ and decreasing when $x > c$. $f(c)$ is a local min if f is decreasing when $x < c$ and increasing when $x > c$. Analytically, $f'(x)$ changes sign from positive to negative on either side of c when $f(c)$ is a local max.

$f'(c) = 0$, $f(c)$ local max

f increases, $f'(x) > 0$ $\qquad$ c $\qquad$ f decreases, $f'(x) < 0$

And $f'(x)$ changes sign from negative to positive on either side of c when $f(c)$ is a local min.

$f'(c) = 0$, $f(c)$ local min

f decreases, $f'(x) < 0$ $\qquad$ c $\qquad$ f increases, $f'(x) > 0$

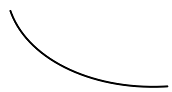
Thus, the FDT shows whether $f(c)$ is a local max or min by the sign of $f'(c)$ on either side of c.

FDT: If $f'(c) = 0$ and $f'(x)$ changes sign from positive to negative at c, then $f(c)$ is a local maximum of f.

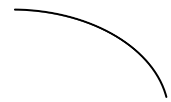
If $f'(c) = 0$ and $f'(x)$ changes sign from negative to positive at c, then $f(c)$ is a local minimum of f.

Second derivative test (SDT): Is $f(c)$ a local max or min?

Before looking at the SDT, consider the *concavity* of curves. Where a graph's shape is roughly similar to a *cup*, the function is *concave up*. If it is roughly similar to a *frown*, the function is *concave down*. In each of the figures below, the function is decreasing on some interval.



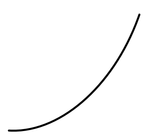
f decreasing, $f'(x)$ is increasing, $f''(x) > 0$



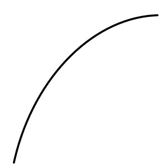
f decreasing, $f'(x)$ is decreasing, $f''(x) < 0$

- The first figure shows a **concave up curve**: f is decreasing but $f'(x)$ is increasing, that is, the slope of a tangent line to the curve becomes less negative from left to right. Thus, $f''(x) > 0$ here.
- The second figure shows a **concave down curve**: f is decreasing and $f'(x)$ is decreasing, that is, the slope of a tangent line to the curve becomes more negative from left to right. Thus, $f''(x) < 0$ here.

The same can be shown for an increasing function. In each of the figures below, the function is decreasing on some interval.



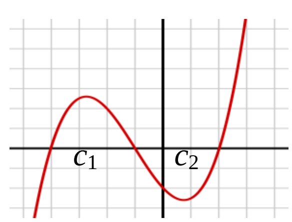
f increasing, $f'(x)$ is increasing, $f''(x) > 0$



f increasing, $f'(x)$ is decreasing, $f''(x) < 0$

- The first figure again shows a **concave up curve**: Here, f is increasing and $f'(x)$ is increasing, that is, the slope of a tangent line to the curve becomes more positive from left to right. Thus, $f''(x) > 0$ here.
- The second figure shows a **concave down curve**: Here, f is increasing but $f'(x)$ is decreasing, that is, the slope of a tangent line to the curve becomes less positive from left to right. Thus, $f''(x) < 0$ here.

To connect concavity to determine of whether a function has $f(c)$ is a local max, a local min at c, the second derivative test (SDT) can be a little quicker than the FDT. Consider the curve:



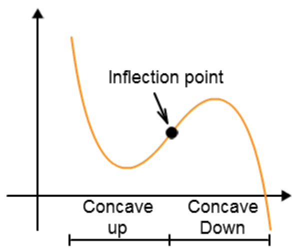
The curve appears to have a local maximum at c_1 and a local minimum at c_2. In fact, c_1 is in an interval where f is concave down, hence, $f''(c) < 0$. C_2 is in an interval where f is concave up, hence, $f''(c) > 0$. We have this:

SDT: If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a *local min* because the graph is concave up there.

If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a *local max* because the graph is concave down there.

An increasing (or decreasing) function may *change concavity* on an interval. Then the second derivative will change sign.

- An *inflection point* is a point where a graph's concavity changes.



If $f''(c) = 0$, $f(c)$ *could* be an inflection point, but not necessarily. (In the examples that follow, we see that $f(x) = x^4$ has both a first and second derivative of zero at $x = 0$, yet $f(0)$ is a minimum of the function.) Thus, to determine the situation in the case of $f''(x) = 0$:

EITHER

1. Resort to the *first derivative test*, checking values on either side of c to see if $f'(c)$ changes sign.

   If it does, we have a local extreme at $x = c$.

   If it doesn't, we have an inflection point at $x = c$.

OR

2. Stay with the *second derivative test*, testing values either side of c to see if $f''(x)$ changes sign.

   If $f''(x)$ *changes sign* at c, then c is an *inflection* point, since concavity has changed.

   If $f''(x) > 0$ on *both sides* then $x = c$ is a local min.

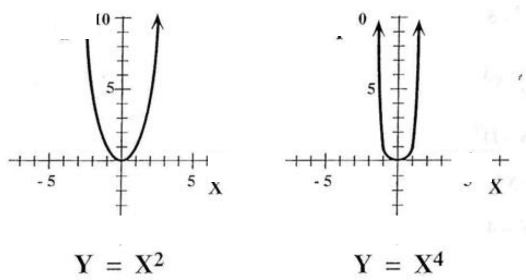
   If $f''(x) < 0$ on *both* sides, then $x = c$ is a local max.

Examples

Power functions are illustrative, because the sketches are simple to inspect.

Examples 1 and 2

$f(x) = x^2$ (quadratic) $f(x) = x^4$ (quartic)



Example 1 $f(x) = x^2$; $f'(x) = 2x = 0$ at $x = 0$, so this is the critical number: $c = 0$.

FDT: $f'(x) = 2x = 0$ at $x = 0$. $f'(-1) = -2 < 0$; f is decreasing left of zero. $f'(1) = 2 > 0$; f is increasing right of zero. $f(0)$ is a local min.

SDT: $f''(x) = 2 > 0$ for all x, so f is concave up everywhere. Thus, $f(0)$ is a local min, as the graph shows.

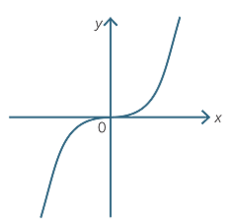
Example 2 $f(x) = x^4$ (quartic); $f'(x) = 4x^3 = 0$ at $x = 0$; $f''(x) = 12x^2 = 0$ at $x = 0$.

What kind of critical point is $c = 0$? There are two ways to find out:

FDT: Checking values into $f'(x)$ on either side of 0, $f'(-1) = 4(-1)^3 = -4$ and $f'(1) = 4(1)^3 = 4$. Because f' changes sign, negative to positive, $c = 0$ is a local min.

SDT: Checking values of $f''(x)$ on either side of 0, $f''(-1) = 12(-1)^2 = 12$ and $f''(1) = 12(1)^2 = 12$. No change in sign, f'' is positive on either side, so the function is concave up, and $c = 0$ is a local min, as the graph shows.

Example 3 $f(x) = x^3$ (cubic)



$f'(x) = 3x^2 = 0$ at $x = 0$; $f''(x) = 6x = 0$ at $x = 0$ also.

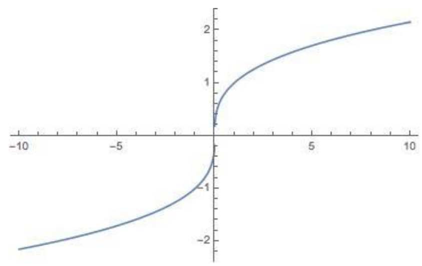
What kind of critical point is $c = 0$? There are two ways to find out:

FDT: Checking values of $f'(x)$ on either side of 0, $f'(-1) = 3(-1)^2 = 3$, and $f'(1) = 3(1)^2 = 3$. Thus, the function is increasing on either side of $c = 0$, so c is an inflection point, as the graph shows.

SDT: Checking values of $f''(x)$ on either side of 0, $f''(-1) = 6(-1) = -6 < 0$. $f''(1) = 6(1) = 6 > 0$. The change in sign indicates the graph is concave down to the left of $x = 0$ and concave up to the right of it, at $x = 0$ is an inflection.

Root functions are also interesting and informative for honing techniques of curve sketching.

Example 4 $\quad f(x) = x^{1/3}$ (cube root)



$f'(x) = \dfrac{1}{3x^{2/3}}$; it's clear that $f'(0)$ does not exist (division by zero). In the DNE sense, $c = 0$ is a critical number of the function. (The tangent line to the function at $x = 0$ is a vertical line.)

FDT: The function is increasing everywhere, as is easily seen in the graph; algebraically, $f'(x)$ is *positive* everywhere it is defined, as $x^{2/3}$ is the square of a cube root. Check $f'(-1)$ and $f'(1)$.

Thus, by the first derivative test, the function is everywhere increasing. There is no local max or min.

Is $x = 0$ an inflection point? $\quad f''(x) = -\dfrac{2}{9x^{5/3}}$

SDT: $f''(x)$ DNE at $x = 0$, for the same reason (division by zero). Checking values of $f''(x)$ on either side of 0:

$f''(-1) = -\dfrac{2}{9(-1)^{5/3}} = \dfrac{2}{9} > 0$ (concave up).

$f''(1) = -\dfrac{2}{9} < 0$ (concave down). $x = 0$ is a point of inflection, as the graph shows.

Example 5 (you investigate) Consider the function $f(x) = |x^2 - 1|$ on $\mathbb{R}$.
Use Desmos Online Graphing Calculator to graph it. Look at it for a while. Draw the graph yourself on paper. Write out the piecewise function, its first and second derivatives (which are also piecewise functions) and find critical points.
Investigate extremes, intervals of increasing and decreasing, concave up and concave down, any inflection points, and so on. An interesting conclusion you should be able to draw (and draw with your pencil): A cusp can be a point of inflection.

Example 6: *Consider the function* $f(x) = x^{2/3}$. Proceed as in Example 5.