Questions from several previous Math 220 finals

1. (25 points) Evaluate each of the following limits. If a particular limit does not exist, write DNE. Show work when the answer is not obvious.

a) $\lim\limits_{x\to-2} \frac{-2x-4}{x^3+2x^2}$: First, plugging in $x=-2$ gives I.F. $\frac{0}{0}$

Do algebra: $\frac{-2(x+2)}{x^2(x+2)} = \frac{-2}{x^2}$. Then, $\lim\limits_{x\to-2} \frac{-2}{x^2} = \frac{-2}{4} = \boxed{-\frac{1}{2}}$

b) $\lim\limits_{x\to-1} \frac{\sqrt{x^2+8}-3}{x+1}$ First, plugging in $x=-1$ gives $\frac{0}{0}$, I.F.

Do algebra: (multiply by conjugate)

$\frac{\sqrt{x^2+8}-3}{x+1} \cdot \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3} = \frac{x^2+8-9}{(x+1)\sqrt{x^2+8}+3} = \frac{x^2-1}{} = \frac{\cancel{(x+1)}(x-1)}{\cancel{(x+1)}\sqrt{x^2+8}+}$

Then, $\lim\limits_{x\to-1} \frac{x-1}{\sqrt{x^2+8}+3} = \frac{-2}{3+3} = \boxed{-\frac{2}{9}}$

c) $\lim\limits_{x\to\infty} \frac{-2x^3-2x+3}{3x^3+3x^2-5x}$ Clearly $\frac{-\infty}{\infty}$ is I.F.

First, this is a limit at ∞ of a rational fcn. and there's a shortcut to find The limit would give the HA, $\lim\limits_{x\to\pm\infty} \frac{P(x)}{Q(x)}$, namely, if $\deg P < \deg Q$, $\lim\limits_{x\to\pm\infty} P/Q = 0$; if $\deg P < \deg Q$, $\lim\limits_{x\to\pm\infty} P/Q = 0$. But if $\deg P = \deg Q$, then $\lim\limits_{x\to\pm\infty} P/Q =$ ratio of lead coeffs.

However, you must show your work for this to get full credit. This entails dividing each term by x^n, where n is highest power of x. Thus:

$\lim\limits_{x\to\infty} \frac{-2x^3/x^3 - 2x/x^3 + 3/x^3}{3x^3/x^3 + 3x^2/x^3 - 5x/x^3} = \lim\limits_{x\to\infty} \frac{-2-2/x^2+3/x^3}{3+3/x-5/x^2}$

$= \frac{-2-0+0}{3+0-0} = \boxed{-\frac{2}{3}}$

d) $\lim\limits_{x\to0^+} \ln x$

$= -\infty$ by graph (must show)

e) $\lim\limits_{x\to5} \frac{1-2x}{(x-5)^2}$

There is a VA at $x=5$. So $\lim\limits_{x\to5} f(x)$ is either ∞ or $-\infty$ or both if LHL $\neq$ RHL. Do a sign analys

LHL: $\lim\limits_{x\to5^-} \frac{1-2x}{x-5} = \frac{-9}{\text{"small" negative}} = +\infty$

RHL: $\lim\limits_{x\to5^+} \frac{1-2x}{x-5} = \frac{-9}{\text{"small" positive}} = -\infty$

Since LHL $\neq$ RHL, $\lim\limits_{x\to5} f(x)$ DNE.

2. (30 points) Find the indicated derivative for each of the following functions. DO NOT SIMPLIFY your answers.

a) $f(x) = 5x^3 - 2x^2 + 19x - 7 + \frac{2}{x^6}$

$f'(x) = 15x^2 - 4x + 19 - 0 - 12x^{-7}$

Power rule: $y = x^n$, $y' = nx^{n-1}$

b) $g(x) = 7^x + \log_7 x$

$g'(x) = 7^x \ln 7 + \frac{1}{x \ln 7}$

Exp and log rules:

$y = a^x$, $y' = a^x \ln a$; $y = \log_a x$, $y' = \frac{1}{x \ln a}$

c) $h(x) = \sqrt[3]{10x^3 - 21x - 5}$ $= (10x^3 - 21x - 5)^{1/3}$

$h'(x) = \frac{1}{3}(10x^3 - 21x - 5)^{-2/3}(30x - 21)$

Chain rule: $y = f(x)^n$, $y' = n[f(x)]^{n-1} \cdot f'(x)$

d) $j(x) = (3x^2 - x + 1)(e^x + 10)$

$j'(x) = (6x - 1)(e^x + 10) + (3x^2 - x + 1)(e^x)$

Product rule: $y = f(x) \cdot g(x)$, $y' = f'(x)g(x) + g'(x)f(x)$

e) $k(x) = \frac{\sqrt{x}+2}{x^8+5}$

$k'(x) = \frac{\frac{1}{2\sqrt{x}}(x^8+5) - (\sqrt{x}+2)8x^7}{(x^8+5)^2}$

Quotient rule: $y = f(x)/g(x)$, $y' = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$

f) $f(x, y) = y^2 \ln(x^2 + 1) - 5y\, e^y$

$f_x = y^2 \cdot \frac{1}{x^2+1} \cdot 2x - 0$

$f_y = 2y \cdot \ln(x^2+1) - (\underbrace{5 \cdot e^y + 5y \cdot e^y}_{\text{product rule}})$

Partial derivatives: $z = f(x, y)$, z_x: hold y term constant

z_y: hold x term constant

3. (30 points) Integrate the following expressions as indicated.

a)

$$\int\left(2x^3-5x+7-\frac{3}{x}+\frac{4}{x^2}\right)dx \;=\; \boxed{\frac{2x^4}{4}-\frac{5x^2}{2}+7x-3\ln|x|-4x^{-1}}$$

Rule: $\int \frac{dx}{x} = \ln|x| + C$

b) u-sub:

$$\int \frac{9x^2}{\sqrt{1-x^3}}\,dx$$

Let $u = 1-x^3$, then $du = -3x^2\,dx$, so $\frac{du}{-3} = x^2\,dx$

$$= 9\int\frac{x^2}{\sqrt{1-x^3}}\,dx = 9\int\frac{du}{\sqrt{u}} = 9\int u^{-1/2}\,du = 9\frac{u^{1/2}}{1/2}+C = \frac{9(1-x^3)^{1/2}}{1/2}+C$$

$$= \boxed{18(1-x^3)^{1/2}+C}$$

Skip c) IBP - Since u-sub doesn't work.

$$\int 6x\,e^{2x}\,dx$$

let $u = 6x$ & let $dv = e^{2x}\,dx$

then $du = 6\,dx$ & $v = \int dv = \int e^{2x}\,dx = \frac{e^{2x}}{2}+C$

Hence $\int 6xe^{2x}\,dx = (6x)\left(\frac{e^{2x}}{2}\right) - \int\frac{e^{2x}}{2}\cdot 6\,dx = \boxed{3xe^{2x} - 3\cdot\frac{e^{2x}}{2} + C}$

d)

$$\int_4^9\left(\frac{x\sqrt{x}+\sqrt{x}}{x^2}\right)dx$$

Simplify the integrand first:

$$\int_4^9 \frac{x\cdot x^{1/2}}{x^2}+\frac{x^{1/2}}{x^2}\,dx = \int_4^9 x^{-1/2}+x^{-3/2}\,dx = \left.\frac{x^{1/2}}{1/2}+\frac{x^{-1/2}}{-1/2}\right|_4^9$$

$$= \left[(2)(9)^{1/2}+(-2)(9)^{-1/2} - \left((2)(4^{1/2})+(-2)(4)^{-1/2}\right)\right]$$

$$= 2\cdot 3 - 2/3 - 2\cdot 2 + 2/2 = 6-\frac{2}{3}-4+1$$

$$= 3-\frac{2}{3} = \boxed{2\frac{1}{3}}$$

e)

$$\int 2x^3(x^2+9)^6\,dx$$

"double" u-sub, where $u = x^2+9$, so $du = 2x\,dx$

and $2x^3 = 2x\cdot x^2$, so u-sub of $x^2 = u-9$

gives $2x^3 = (u-9)\,du$, hence $\int u^6\cdot u-9\cdot du = \int u^7 - 9u^6\,du$

Since $2x\,dx = dx$

$$= \frac{u^8}{8} - \frac{9u^7}{7} + C = \boxed{\frac{(x^2+9)^8}{8} - \frac{9(x^2+9)}{7} + C}$$

Though curve-sketching is not on the final, concepts of increasing / decreasing / critical #s / concavity / inflection are ~~still~~ included.

→ Degree $n=4$, Lead coeff >0, Ends ↖ ↗

4. (25 points) Suppose $f(x) = x(x-4)^3$. The first and second derivatives of f are:
$f'(x) = 4(x-4)^2(x-1)$ and $f''(x) = 12(x-4)(x-2)$.
Do not waste time verifying these derivatives. They are correct. Additionally, the graph of f does not contain any horizontal nor vertical asymptotes, so do not attempt to find any.

a) Find ordered pairs for the x and y intercepts on the graph of f.

$f(0) = 0(0-4)^3 = 0$: $\boxed{\begin{array}{l} y\text{-int } (0,0) \\ x\text{-int } (4,0) \text{ and } (0,0) \end{array}}$

$f(x) = 0 = x(x-4)^3$;
so $x=0$, $x=4$

b) Find intervals where f is increasing and where f is decreasing.

$f\uparrow$ when $f'>0$
$f\downarrow$ when $f'<0$

Mark off zeroes of $f'(x)$, that is, critical numbers $c=1,4$

$f'(x) > 0$: $\underset{f\downarrow}{f'<0}$ 1 $\underset{f\uparrow}{f'>0}$ 4 $\underset{f\uparrow}{f'>0}$

$\boxed{\begin{array}{l} f\uparrow \text{ on } (1,4)\cup(4,\infty) \\ f\downarrow \text{ on } (-\infty,1) \end{array}}$

Since $f(4)$ exists, write $f\uparrow$ on $(1,\infty)$

c) Find ordered pairs for any local extrema.

Local min : $(1, f(1)) = (1,-27)$

No local max, since $f\uparrow$ from 1 to ∞

Alternately, $f''(1) > 0$, so $f(x)$ is c. up. on interval around $x=1$, hence $f(1)$ is a local min.

d) Find intervals where f is concave upward and where f is concave downward.

f is c. up where $f''>0$; f is c. down where $f''<0$

Mark off zeroes of $f''(x)$

$\underset{\cup}{f''>0}$ 2 $\underset{\cap}{f''<0}$ 4 $\underset{\cup}{f''>0}$;

Concave up on $(-\infty,2)\cup(4,\infty)$
Concave down ~~on~~ $(2,4)$

e) Find ordered pairs for any inflection points.

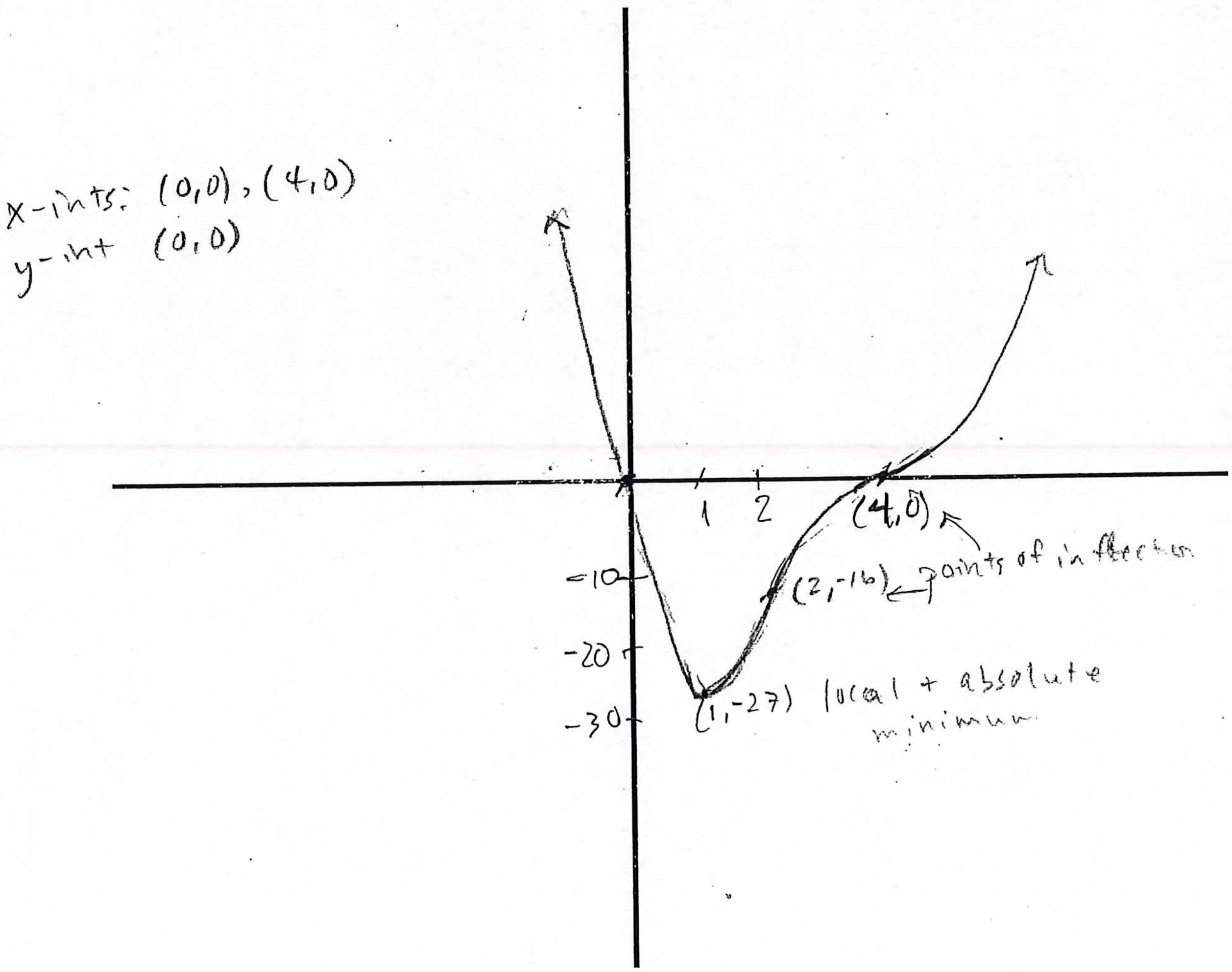
f changes concavity at $x=2$ and $x=4$

$f(2) = -16$, $f(4) = 0$

Inflection pts $(2,-16)$ and $(4,0)$

f) Use everything found in parts a) through e) to sketch the graph of f below. Your graph should be as accurate as possible.

GRAPH FOR PROBLEM 4



5. (15 points) At home games for the Binghamton Rumble Ponies, hot dogs are sold for \$3 apiece. At that price, 600 hot dogs per game are typically sold. For each decrease in the price of hot dogs by \$0.50, it is determined that the number of hot dogs sold would increase by 200. The cost of selling x hot dogs is given by the function $C(x) = 0.5x + 300$

a) Write an expression for p, the price per hot dog as a function of x, the quantity of hot dogs sold.

The price fcn. $p(x)$ is linear, having two known points $(600, 3)$ and $(800, 2.50)$. Using point-slope form to find $p(x)$: $y - y_1 = m(x - x_1)$ where $m = \Delta y / \Delta x$.

Here, $y = p(x)$. $m = \frac{3 - 2.50}{600 - 800} = \frac{+.5}{-200}$ or $\frac{-5}{2000} = -\frac{1}{400}$

Thus, $p - 3 = -\frac{1}{400}(x - 600)$ is the line, and (see below ~~p(x)...~~)

b) Determine $P(x)$, the profit generated on hot dog sales as a function of the quantity sold.

$P(x) = R(x) - C(x)$ where $R(x) = x \cdot p(x)$

$P(x) = x\left(-\frac{1}{400}x + 4\frac{1}{2}\right) - (0.5x + 300)$

Simplifying: $\boxed{P(x) = \frac{-x^2}{400} + 4x - 300}$ profit fcn.

c) What is the maximum possible profit the Rumble Ponies can make on the sale of hot dogs on a given night?

Set $P'(x) = 0$ to find critical #.

$P'(x) = \frac{-2x}{400} + 4 = 0$, $\frac{2x}{400} = 4$, $\boxed{x = \frac{1600}{2} = 800 \text{ hot dogs}}$

[Notice $P'' = -\frac{2}{400} < 0$, so $P(800)$ is max.]

Expression for price as fcn of quantity

* $p = -\frac{1}{400}(x - 600) + 3 = -\frac{1}{400}x + \frac{600}{400} + 3 = \boxed{-\frac{1}{400}x + 4\frac{1}{2}}$

e.g. $p(800 \text{ hot dogs}) = -2 + 4\frac{1}{2} = \$2.50$, as required.

6. (16 points)

Based on past experience, a company has determined that its sales revenue R (in dollars) is related to its advertising according to the formula $R(x, y) = 6000x + y^2 + 4xy$, where x is the amount spent on radio advertising and y is the amount spent on television advertising. If the company plans to spend $30,000 on these two means of advertising, how much should it spend on each method to maximize its sales revenue?

I) Using substitution of the constraint $x + y = 30,000$ into the objective fcn $R(x,y)$ to reduce it to $R(x)$ (single-variable fcn) gives:

$$R(x) = 6000x + (30,000 - x)^2 + 4x(30,000 - x)$$

Then $$R'(x) = 6000 + \underbrace{2(30,000 - x)(-1)}_{\text{chain rule}} + 120,000 - 8x$$

$$= 6000 + {}^{-}60,000 + 2x + 120,000 - 8x$$

$$= 66,000 - 6x = 0$$

$$\boxed{\begin{aligned} x &= \$11,000 \\ y &= \$19,000 \end{aligned}}$$

II) Lagrange set-up: $F(x, y, \lambda) = R(x,y) - \lambda \cdot g(x,y)$ (where $g(x,y) = x + y - 30,000$) gives system of partials

$$\begin{cases} F_x = 6000 + 4y - \lambda = 0 \\ F_y = 2y + 4x - \lambda = 0 \\ F_\lambda = x + y - 30,000 = 0 \end{cases}$$

Solving for x in terms of y by eliminating λ in F_x, F_y, gives $2y = 4x - 6000$ or $y = 2x - 3000$.

Substituting this into F_λ gives $x + 2x - 3000 - 30,000 = 0$ $\boxed{\begin{aligned} x &= \$11,000 \\ y &= \$19,000 \end{aligned}}$

7. (10 points) Given the equation $x^2 - xy + y^3 = 8$:

a) Find the derivative $\frac{dy}{dx}$. Using implicit differentiation:

$$2x - \left(1 \cdot y + x \cdot \frac{dy}{dx}\right) + 3y^2 \frac{dy}{dx} = 0$$

$$2x - y - x\frac{dy}{dx} + 3y^2 \frac{dy}{dx} = 0$$

$$2x - y = (x - 3y^2)\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{2x-y}{x-3y^2}}$$

b) Find the equation of the tangent line to the curve at the point (0, 2). Any form of linear equation is acceptable.

$\frac{dy}{dx} = m_{\text{tangent}}$ along $x^2 - xy + y^3 = 8$

At $(0,2)$, the tangent line has eqn: (point-slope form.

$$y - 2 = \underbrace{\left.\frac{2x-y}{x-3y^2}\right|_{(0,2)}}_{\text{slope, evaluate at } (0,2)} (x - 0)$$

$y - 2 = \frac{-2}{-12}(x - 0)$ or $y - 2 = \frac{1}{6}x$

that is, $\boxed{y = \frac{1}{6}x + 2}$

8. (15 points) Suppose the quantity q of a certain commodity changes depending on the price p, in dollars, as given by the equation $q = 50 - \frac{p}{3}$ where $0 \le p \le 150$.

a) Find the elasticity of demand function, $E(p)$.

$$E(p) = -\frac{p}{q} \cdot q' = \frac{-p}{50 - p/3} \cdot \frac{-1}{3} = \boxed{\frac{p}{150 - p}}$$

b) Find $E(80)$. Write a sentence explaining the real-world meaning of your answer in terms of the relationship between price and revenue when $p = 80$.

$E(80) = \frac{80}{150-80} = \frac{80}{70}$, $E(80) > 1$ so the commodity is elastic at this price, hence a small increase in price above \$80 will cause revenue to decrease.

c) Use $E(p)$ to determine the maximum possible revenue. *

$R'(p) = 0$ when $E(p) = 1$ from $R'(p) = q(1 - E)$

Hence, setting $E(p) = 1$ gives $\frac{p}{150 - p} = 1$, or

$p = 150 - p$

$2p = 150$

$\boxed{p = \$75}$ This is the price that ensures max revenue, which is ...

... $R(p) = p \cdot q(p)$]

$R(75) = 75(50 - 75/\ ) = 75(25) = \boxed{\$1875}$ max revenue

* Find $R(p)$ for this problem + graph it.

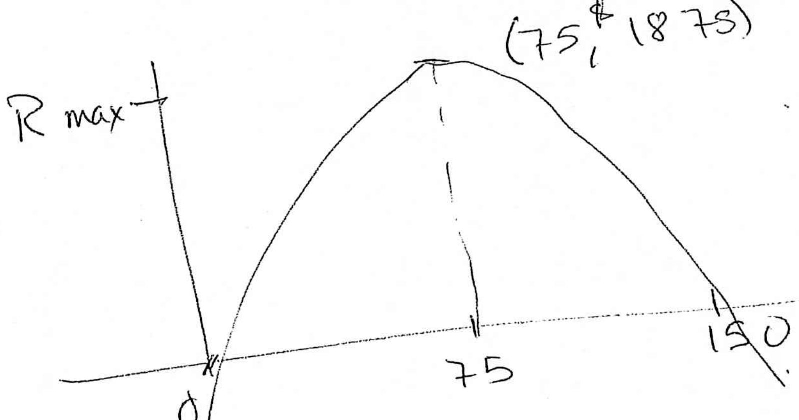
$R(p) = p \cdot q(p) = p\left(50 - \frac{p}{3}\right)$; a parabola with negative leading coefficient: $R = 50p - \frac{p^2}{3}$

It's easier to graph in factored form:

$R(p) = p\left(50 - \frac{p}{3}\right) = 0$ when $p = 0, 150$.

$R'(p) = 50 - \frac{2p}{3} = 0$ when $p = \frac{150}{2} = \$75,$

as found by setting $E(p) = 1$.



Find $R(75)$ to scale the curve;

$R(75) = 75\left(50 - \frac{75}{3}\right) = 75(50-25) = 75(25)$

Max revenue $R = \$1875$

9. (18 points) The following problems deal with money growth. The parts are not related to one another. You DO NOT NEED TO SIMPLIFY your answers.

a) An investor wants to have \$20,000 at the end of 15 years, and is able to earn an annual interest rate of 10%, where interest is compounded monthly. How much money will need to be invested to guarantee this outcome?

$F = P(1+\frac{r}{n})^{nt}$, solve for P(principal)

$20{,}000 = P(1+\frac{.1}{12})^{(12)(15)}$

$$\boxed{P = \frac{20{,}000}{(1+\frac{.1}{12})^{(12)(15)}}}$$

b) Suppose you have \$1,000, and want to invest this money until it reaches \$2,000. How long will it take to reach your goal if you invest in a 5% interest-bearing account, compounded continuously?

Doubling time for cts. cmpd. is given by

$2P = Pe^{rt} \longrightarrow 2 = e^{rt}$

Solving for time t: $\ln 2 = \ln e^{rt} = rt \ln e = rt$

That is, $t = \ln 2/r$ or here, $\boxed{t = \ln 2/.05}$

c) Starting now, I plan to transfer money continuously into my Roth IRA at the constant rate of \$5,500 per year, the maximum amount allowed by the Internal Revenue Service. The account grows at the annual rate of 9%, compounded continuously. Assuming I plan to retire in twenty years, how much will be in the account when I retire?

Integration is required. The formula for final value of an income stream (money flow) is

$FV = e^{rT}\int_0^T f(t)e^{-rt}\,dt$, where $\boxed{\begin{array}{l} T = 20 \\ r = .09 \\ f(t) = 5500 \end{array}}$

$$\boxed{FV = e^{(.09)(20)}\int_0^{20} 5500e^{-.09t}\,dt}$$

$$FV = e^{1.8} \cdot 5500 \int_0^{20} e^{-.09t}\,dt$$

Notice that T is a given value, but t is the variable of integration

From $\int e^{ax}\,dx = \dfrac{e^{ax}}{a} + C$

we get:

$$FV = e^{1.8} \cdot 5500 \left[\frac{e^{-.09t}}{-.09} \right]_0^{20}$$

$$= \frac{5500}{-.09} e^{1.8} \left[e^{-.09(20)} - e^{0} \right]$$

Since $-.09$ is not affected by variable, it can come outside the "F(b) − F(a)" calculation. Distribute $e^{1.8}$ to get:

$$= \frac{-5500}{.09} \left[e^{1.8} e^{-1.8} - e^{1.8} \cdot \underbrace{e^{0}}_{1} \right]$$

$$FV = \frac{-5500}{.09} \left[\underbrace{e^{0}}_{1} - e^{1.8} \right] = \frac{-5500}{.09} \left[1 - e^{1.8} \right]$$

If we use a calculator, we find

FV = \$30,859 value of the IRA in 20 yrs

Notice the two formulas for continuous income stream calculations are related:

$$\boxed{\begin{aligned} FV &= e^{rT} \int_0^T f(t)\, e^{-rt}\, dt \\ PV &= \int_0^T f(t)\, e^{-rt}\, dt \end{aligned}}$$

memorize these!

So $FV = e^{rT} \cdot PV$ or $PV = e^{-rT} \cdot FV$

This reminds us of the Ch. 5 formulas for compound interest:

$\boxed{F = Pe^{rt}}$ and $\boxed{P = Fe^{-rt}}$

memorize these, too!

d) I decide to start an online business selling hats. I have determined that the rate at which money will flow continuously into the account is given by the function $f(t) = 2000 - 20t^2$, and that the business will be viable for a period of 10 years. Income earned from the business can be invested in a bank account earning 6% annual interest. Set up, but do not solve, an integral that gives the fair-market value of this business (how much the business is worth *right now*).

$$PV = \int_0^T f(t)\, e^{-rt}\, dt$$

present value formula for an income stream

$$\boxed{PV = \int_0^{10} (2000 - 20t^2)\, e^{-.06t}\, dt}$$

This integral requires integration by parts!
Leave as integral on exam, unless f(t) is consta

10. (16 points)

The value of a new stock has changed a lot since it came on the market nine months ago. The function $V(t) = \frac{1}{3}t^3 - \frac{9}{2}t^2 + 18t + 15$ gives the value (in dollars) of the stock t months after it first started. Answer the following questions. Put units of measure on all of your answers.

a) What is the current value of the stock?

$t = 0$

$\boxed{V(0) = 15}$, so the stock is worth \$15 "now"

b) What was the average value of the stock over the nine-month interval? You don't have to simplify this answer (it's ugly).

V_{avg} on $[0,9] = \frac{1}{9-0}\int_0^9 V(t)\,dt = \frac{1}{9}\int_0^9 \left(\frac{1}{3}t^3 - \frac{9}{2}t^2 + 18t + 15\right) dt$

$= \frac{1}{9}\left[\frac{t^4}{12} - \frac{9t^3}{6} + \frac{18t^2}{2} + 15t\right]_0^9 = \frac{1}{9}\left[\frac{9^4}{12} - \frac{9\cdot 9^3}{6} + 9\cdot 9^2 + 15\cdot\right.$

c) At the five-month mark, was the stock's value increasing or decreasing? Use calculus to justify your answer.

This asks what the sign of $V'(5)$ is.

$V' = t^2 - 9t + 18$, $V'(5) = 25 - 45 + 18 = -2 < 0$

So Value $\boxed{V \text{ is decreasing}}$ at the 5-month mark.

d) At what time during its nine-month existence did the stock have its highest value? Use calculus to justify your answer.

$V(t)$ is cubic with positive lead coefficient

$V'(t) = t^2 - 9t + 18 = 0$ at $\boxed{t = 6, 3}$ crit. numbers

$(t-6)(t-3) = 0$

By SDT, we find $V''(t) = 2t - 9$ is positive at $t = 6$

...tive at $t = 3$. Hence $V(3)$ is ~~the~~ local max on interval [0, 9]; and $V(6)$ is ~~the~~ local min on the interval.

(Recall that a fcn is concave up on intervals where the second derivative is greater than zero, which means that a critical point in the interval is a minimum. Conversely, a critical point in a concave down interval, where the second derivative is negative is a maximum. ~~That is, where~~

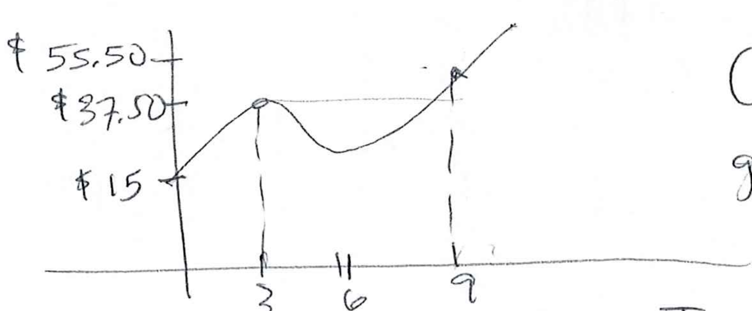
So here $V''(3) < 0$, so $V(3)$ is a local max.

<u>However</u> — we still have to check the endpts of the time interval [0, 9].

$V(0) = \$15$ to start $\qquad V(3) = \$37.5 \qquad \boxed{V(9) = \$55.5}$

$\uparrow$ local max $\qquad \uparrow$ abs. max

Thus, the maximum value of the stock in this time interval occurs at the end of the 9 months.



Check $V(6)$ to see if the graph is accurate with respect to the local min.

Is $V(0)$ the abs min, or is $V(6)$?

11. (25 points)

rate fcn.

Bob is riding his bicycle along a straight path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute.

a) Find the value of the definite integral:

$$\int_0^{10} B(t)dt = \int_0^{10} t^3 - 6t^2 + 300\,dt = \left[\frac{t^4}{4} - \frac{6t^3}{3} + 300t\right]_0^{10}$$

$$= \frac{10{,}000}{4} - 2(1000) + 300(10) = \boxed{3500 \text{ meters}}$$

b) Write a sentence that describes the real-world meaning of your answer from part a). Use appropriate units as part of your explanation.

Important

From the fundamental thm. of calculus, $\int_a^b$ rate fcn $f(t)\,dt$ gives the total value of $f(t)$ on $[a,b]$. Thus, Bob biked 3500 m. in 10 min

c) The value of the expression...

$$\frac{1}{5}\int_2^7 B(t)dt$$ Notice $5 = 7-2$ (hint)

...can be best described as which of the following? Circle the one best answer.

(A) The average distance Bob traveled from time $t = 2$ minutes to $t = 7$ minutes.
(B) 1/5 of the total area Bob transversed over the period $t = 2$ to $t = 7$ minutes.
(C) The amount of Bob's total acceleration from $t = 2$ to $t = 7$ minutes.
(D) Bob's average velocity over the 5-minute span from $t = 2$ to $t = 7$ minutes.

Because $B(t)$ avg on $[2,7] = \frac{1}{7-2}\int_2^7 B(t)\,dt$

d) Suppose Bob's initial position (at time $t = 0$) on the path was 20 meters to the right of a large tree that people often use as a landmark. Find an expression for Bob's position relative to the tree at any time t, where $0 \le t \le 10$.

$B(t)$ = velocity, so position $s(t) = \int B(t)\,dt$

$s(t) = \int t^3 - 6t^2 + 300\,dt = \frac{t^4}{4} - \frac{6t^3}{3} + 300t + C$ $\boxed{C = 20}$

$s(0) = 20$, that is, $0 - 0 + 0 + C = 20$

e) At what time t was Bob's acceleration smallest? Give a full justification of your answer.

Since velocity $= B(t)$, acceleration $= B'(t)$.

$B'(t) = 3t^2 - 12t = 3t(t-4) = 0$, $t = 0, 4$.

$B''(t) = 6t - 12$; $B''(0) = -12 < 0$, c. down. $B(0)$ maximum

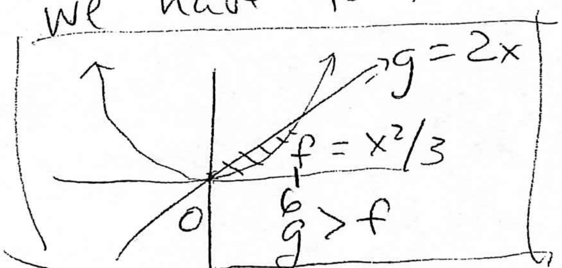
$B''(4) = 12 > 0$, c. up. $\boxed{B(4) \text{ minimum}}$

12. ~~8.~~ (20 points) The following questions involve area. Part a) and Part b) are unrelated.

a) Suppose $f(x) = \frac{x^2}{3}$ and $g(x) = 2x$. Find the area of the region bounded by these curves.

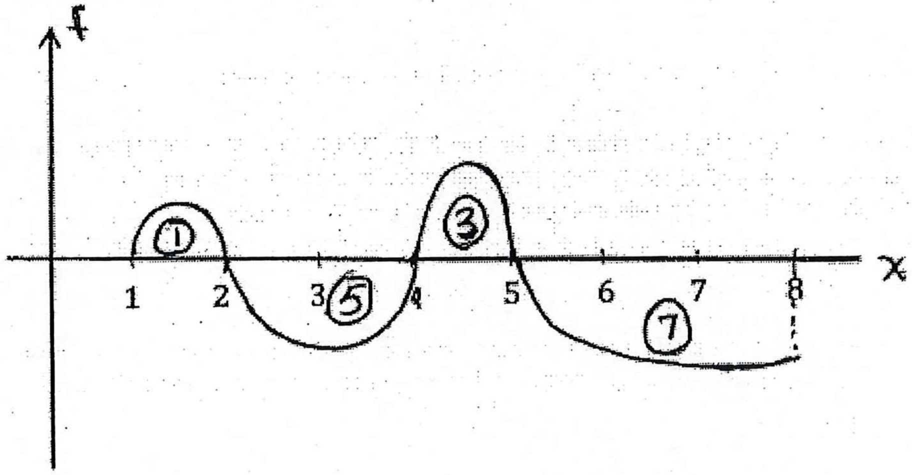
Area = $\int_a^b (\text{top fcn} - \text{bottom fcn})\,dx$ So we need to sketch f & g to determine which is greater. Also, we have to find where $f = g$ to get a & b:

$$2x = \frac{x^2}{3} \longrightarrow x^2 - 6x = 0, \quad x(x-6) = 0$$



$$\int_0^6 \left(2x - \frac{x^2}{3}\right)dx = \left[x^2 - \frac{x^3}{9}\right]_0^6 = \boxed{12}$$

b) Consider the graph of f (not drawn to scale) below, defined on $[1, 8]$. The circled numbers represent the AREAS of the regions between the graph of f and the x-axis. Find the value of each of the definite integrals below:



$\int_1^8 f(x)dx =$ -8 ($1-5+3-7$)

$\int_2^5 f(x)dx =$ -2 ($-5+3$)

$\int_8^4 f(x)dx =$ 4 ($-\int_4^8 f(x)dx = (3-7)$)

$\int_3^3 f(x)dx =$ 0

$\left|\int_1^5 f(x)dx\right| =$ 1 ($|1-5+3|$)

$\int_1^4 |f(x)|\,dx =$ 6 ($1+5$)

Know how to graph <u>all</u> essential fcns. as covered and used throughout. Namely:

$y = \ln x$, $y = e^x$, $y = \sqrt{x}$, $y = x^{1/3}$

$y = e^{-x}$, $y = x^2$, $y = (x \pm a)^2 \pm b$, $y = x$, $y = -x$
(shifted parabola)

$y = \text{constant}$, $x = \text{constant}$, $y = \frac{1}{x}$, $y = \frac{1}{x \pm a}$

$y = |x|$, $y = mx + b$, $y = a_n x^n + a_{n-1} x^{n-1} + \cdots + ax + a_0$

and be able to find x- and y-intercepts, as well as how to find intersection pts. of two curves.