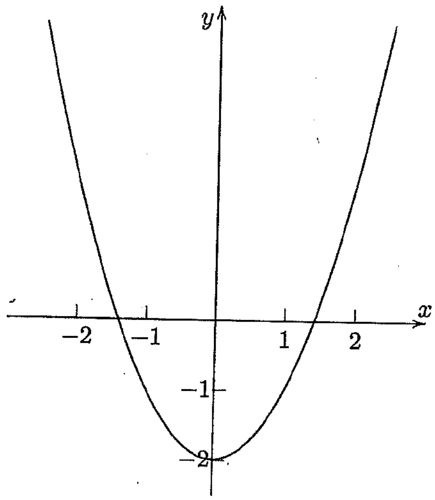
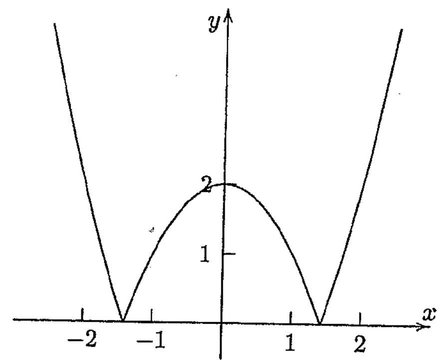
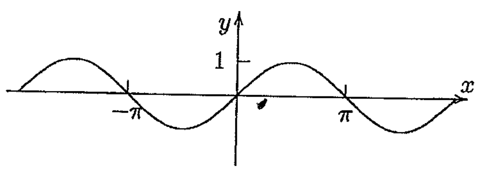


$y = x^2 - 2$

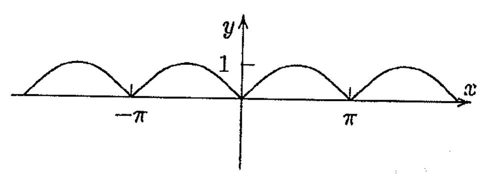


$y = |x^2 - 2|$

memorize



$y = \sin x$



$y = |\sin x|$

7.3 Absolute Value and Inequalities Combined

Important Idea 7.3.1.

1. *If $a > 0$, then $|x| < a$ if and only if $-a < x < a$.*
2. *If $a > 0$, then $|x| > a$ if and only if $[x < -a$ or $x > a]$.*

There are two ways to understand the statements in Important Idea 7.3.1. Both are useful. One should not be ignored in favor of the other.

A geometric interpretation is an extension of the geometric definition of absolute value (Definition 7.2.3). We think of $|x|$ as the distance from 0 to x on a numberline. So $|x| < a$ would mean all numbers x whose distance from 0 on a numberline is less than a. This would certainly be all of the numbers between $-a$ and a, in other words: $-a < x < a$. Similarly, $|x| > a$ would be the set of all numbers whose distance from 0 on the numberline is greater than a. Those numbers would be the positive numbers greater than a and the negative numbers less than $-a$, in other words: $[x < -a \ \ or \ \ x > a]$.

The second interpretation is an algebraic one. If $|x| < a$ then we are saying that if we ignore the sign, the *size* of the number x is less than a. Certainly those would be the positive numbers less than a and the negative numbers greater than $-a$, and of course 0. This combines to $-a < x < a$. On the other hand the set of numbers whose *size* is greater than a would be the positive numbers greater than a and the negative numbers less than $-a$. So $|x| > a$ means exactly $[x < -a \ \ or \ \ x > a]$.

Notice that when the sign is "<" we get a single interval result. When the sign is ">" we get a disjoint set. This should make sense. For "greater than" we are looking at extreme values. For "less than" we are looking to set a boundary.

Geometrically then, we could interpret $|x| < 4$ as being all of the numbers on the number line whose distance from the origin is less than 4. This is the interval $(-4,\ 4)$. Algebraically we think of it as all of the numbers whose *size* is less than 4. This translates to $-4 < x < 4$.

Given $|x| > 4$ we think of all of those numbers on the number line whose distance from the origin is greater than 4. These would be the numbers on the ends of the number line, those lying beyond 4 to the right and beyond -4 to the left. This is the set $(-\infty, -4) \cup (4, \infty)$. Algebraically we think of $|x| > 4$ as all of the numbers whose *size* is greater than 4. This would be $[x < -4 \ \ or \ \ x > 4]$.

Notice that if we combine the sets from $|x| < 4$ and $|x| > 4$ we get $(-4, 4) \cup (-\infty, -4) \cup (4, \infty)$, which is the entire set of real numbers, except for the numbers -4 and 4. It may seem obvious, but it is worth the observation that $|x| < 4$ and $|x| > 4$ and $|x| = 4$ are disjoint (non-overlapping) sets which together encompass all of $\Re$. Again, you can think of this two ways. Geometrically, we think that the entire number line is covered by the union of these sets. Algebraically we think that every real number is included in one, and only one, of these sets. In other words, for every real number, its absolute value must be less than 4, equal to 4 or greater than 4.

In the previous section's study of absolute value we interpreted $|x - c|$ geometrically to mean "the distance between x and c on the number line." So, $|x + 3| = 5$ meant that the distance between x and -3 on the number line is 5. We solved that to say that x had to be either 2 or -8. We carry this geometric interpretation to inequalities.

Important Idea 7.3.2.

1. *For $a > 0$, $|x - c| < a$ represents the set of numbers x whose distance from c on the number line is less than a.*
2. *For $a > 0$, $|x - c| > a$ represents the set of numbers x whose distance from c on the number line is greater than a.*

Our algebraic interpretations are directly applied from Important Idea 7.3.1: $|x - c| < a$ means $-a < (x - c) < a$, and $|x - c| > a$ translates to $[(x - c) < -a \ \ or \ \ (x - c) > a]$.

Example 7.3.1.

Solve for x: $|x - 3| < 8$

Geometric solution: We want the set of all numbers x whose distance from 3 on the number line is less than 8. So, beginning at 3 we mentally go a distance 8 in either direction to get the solution *set* $(-5, 11)$.

Algebraic solution $|x-3| < 8 \implies -8 < (x-3) < 8 \implies -5 < x < 11$.

Comprehension Check 7.2.

1. *Express in words the geometric interpretation of* $|x+2| > 6$ *and then solve for* x.
2. *Express algebraically* $|x+2| > 6$ *and then solve for* x.
3. *Express as an inequality that uses absolute value: "The set of all numbers* x *whose distance from* 6 *on the number line is at least* 12*".*
4. *Express as an inequality that uses absolute value: "The set of all numbers* x *whose distance on the number line is within* 4 *units of* -9*".*

We will not always have expressions as simple as "$(x-c)$" inside our absolute value signs. Any function of x could be there. Algebraically, we still manipulate this using the idea of Important Idea 7.3.1:

For $a > 0$, $|f(x)| < a$ if and only if $-a < f(x) < a$.
For $a > 0$, $|f(x)| > a$ if and only if $f(x) < -a$ *or* $f(x) > a$.

In the study of calculus there are several places where mathematicians are interested in how close together the graphs of two functions are. For instance, if they want to find the values of x for which the graphs of functions f and g are within some vertical distance c of each other, they will set up the equation $|f(x) - g(x)| < c$. We know that the geometric interpretation of this equation is "the set of numbers x for which the distance between f and g is less than c."

Example 7.3.2.
Solve for x *and give a geometric interpretation:* $|2-5x| < 9$.

We solve this algebraically:

$$|2-5x| < 9$$
$$-9 < 2-5x < 9$$
$$-11 < -5x < 7$$
$$\frac{11}{5} > x > -\frac{7}{5}$$

(note the sign change in the last step as we multiply by a negative number)
So, the solution set is all x *in the interval* $(-\frac{7}{5}, \frac{11}{5})$.

For a geometric interpretation we can think of $f(x) = 2$ *and* $g(x) = 5x$ *and so say that the graphs for the equations* $y = 5x$ *and* $y = 2$ *are within nine (vertical) units for any* x *in the interval* $(-\frac{7}{5}, \frac{11}{5})$.

Actually, our geometric interpretation is not unique. We could think of $f(x) = 2 - 5x$ and $g(x) = 0$ and say that the graphs for $y = 2 - 5x$ and $y = 0$ (the x-axis) are within nine (vertical) units for any x in the interval $(-\frac{7}{5}, \frac{11}{5})$.

Or, we could even think of $f(x) = 2 - 3x$ and $g(x) = 2x$ and say that the graphs for $y = 2 - 3x$ and $y = 2x$ are within nine (vertical) units for any x in the interval $(-\frac{7}{5}, \frac{11}{5})$.

All of these interpretations fit the form $|f(x) - g(x)| < c$ for $c = 9$.

Comprehension Check 7.3.

1. *On the same set of axes, carefully sketch* $f(x) = (3x)$ *and* $g(x) = 5$.
2. *From your graphs, estimate the* x *interval over which values the vertical distance between your graphs is less than* 4.
3. *The inequality that describes this situation is* $|3x - 5| < 4$. *Solve this inequality and compare the result to your estimate in part* (2).

We now look at several examples to reinforce the algebra of absolute values and inequalities.

Example 7.3.3.
Solve for x: $|x^2 - 6x + 9| < 25$.

$$|x^2 - 6x + 9| < 25$$
$$|(x-3)^2| < 25$$
$$|x-3| < 5$$

So, (by mental geometric interpretation), the solution set is $(-2, 8)$.

Example 7.3.4.
Solve for x: $\left|3 - \frac{x}{2}\right| \geq 1$.

$$\left|3 - \frac{x}{2}\right| \geq 1$$
$$2\left|3 - \frac{x}{2}\right| \geq 2 \cdot 1$$
$$|6 - x| \geq 2$$

So, the solution set is $(-\infty, 4] \cup [8, \infty)$.

In Examples 7.3.3 and 7.3.4 we see that the last step is easier to do mentally than to write out the corresponding inequalities $-5 < (x-3) < 5$ and $[(6-x) < -2$ *or* $(6-x) > 2]$ and solve them. In Example 7.3.4 we are also reminded that $|a - b| = |b - a|$.

Example 7.3.5.
Solve for x: $\left|\frac{1}{x} - 1\right| < 3$.

$$\left|\frac{1}{x} - 1\right| < 3$$
$$\left|\frac{1-x}{x}\right| < 3$$
$$-3 < \frac{1-x}{x} < 3$$
$$-3 < \frac{1-x}{x} \quad AND \quad \frac{1-x}{x} < 3$$

Now we need to consider the cases $x > 0$ *and* $x < 0$. *(The domain tells us* $x \neq 0$*).*

If $x > 0$, the last statement can be rewritten as:

$$-3x < 1 - x \text{ AND } 1 - x < 3x$$
$$-2x < 1 \text{ AND } 1 < 4x$$
$$x > -\frac{1}{2} \text{ AND } x > \frac{1}{4}$$

So, if $x > 0$ AND $x > -\frac{1}{2}$ AND $x > \frac{1}{4}$ we simplify to $x > \frac{1}{4}$ is part of the solution to the original problem.

If $x < 0$, we have similar algebra, but must change the sign when multiplying by x:

$$-3x > 1 - x \text{ AND } 1 - x > 3x$$
$$-2x > 1 \text{ AND } 1 > 4x$$
$$x < -\frac{1}{2} \text{ AND } x < \frac{1}{4}$$

So, if $x < 0$ AND $x < -\frac{1}{2}$ AND $x < \frac{1}{4}$ we simplify to $x < -\frac{1}{2}$ is the other part of the solution to the original problem.

Our solution set is $(-\infty, -\frac{1}{2}) \cup (\frac{1}{4}, \infty)$.

Example 7.3.6.

For what values of x are the graphs of the functions $f(x) = \dfrac{x+1}{2}$ and $g(x) = \dfrac{2x-1}{3}$ within one vertical unit of each other?

We need to solve the equation $|f(x) - g(x)| < c$. We use f and g as defined above, and $c = 1$. We get: $\left|\dfrac{x+1}{2} - \dfrac{2x-1}{3}\right| < 1$ and need to solve for x.

$$\left|\frac{x+1}{2} - \frac{2x-1}{3}\right| < 1$$
$$6 \cdot \left|\frac{x+1}{2} - \frac{2x-1}{3}\right| < 6 \cdot 1$$
$$|3(x+1) - 2(2x-1)| < 6$$
$$|-x+5| < 6$$

So, the solution set is $(-1, 11)$.

Example 7.3.7.

Solve for x: $\left|\dfrac{4-5x}{2}\right| \geq 1$

$$\left|\frac{4-5x}{2}\right| \geq 1$$
$$2 \cdot \left|\frac{4-5x}{2}\right| \geq 2 \cdot 1$$
$$|4-5x| \geq 2$$
$$4 - 5x \geq 2 \quad or \quad 4 - 5x \leq -2$$

$$-5x \geq -2 \quad or \quad -5x \leq -6$$
$$x \leq \frac{2}{5} \quad or \quad x \geq \frac{6}{5}$$

So, the solution set is $(-\infty, \frac{2}{5}) \cup (\frac{6}{5}, \infty)$

Example 7.3.8.
Solve for x: $|2x-7| > |2-3x|$

Solution 1: This is true for $x = \frac{2}{3}$, *so include* $\frac{2}{3}$ *in the solution set. Then, for* $x \neq \frac{2}{3}$, *rewrite as* $\frac{|2x-7|}{|2-3x|} > 1$ *and solve in the manner as Example 7.3.5.*

Solution 2: We use the squaring technique:

$$|2x-7| > |2-3x|$$
$$|2x-7|^2 > |2-3x|^2$$
$$4x^2 - 28x + 49 > 4 - 12x + 9x^2$$
$$-5x^2 - 16x + 45 > 0$$
$$(x+5)(-5x+9) > 0$$

x	$(-\infty,-5)$	$\{-5\}$	$(-5,\frac{9}{5})$	$\{\frac{9}{5}\}$	$(\frac{9}{5},\infty)$
$(x+5)$	$-$	0	$+$	$+$	$+$
$(-5x+9)$	$+$	$+$	$+$	0	$-$
$(x+5)(-5x+9)$	$-$	0	$+$	0	$-$

So, the solution set is the interval $(-5, \frac{9}{5})$.

There will be times, particularly in calculus, where it is important to know the absolute value of a function, $|f(x)|$. From the definition of absolute value we can get:

$$|f(x)| = \begin{cases} -f(x) & if \quad f(x) < 0 \\ f(x) & if \quad f(x) \geq 0 \end{cases}$$

So, determining $|f(x)|$ essentially amounts to determining for which values of x the function f is positive and for which values of x the function is negative. Then we just apply the appropriate sign.

Example 7.3.9.
Rewrite $|x^3 - 5x^2 + 4x|$ *as a piecewise defined function.*

$$|x^3-5x^2+4x| = \begin{cases} -(x^3-5x^2+4x) & if \quad (x^3-5x^2+4x) < 0 \\ x^3-5x^2+4x & if \quad (x^3-5x^2+4x) \geq 0 \end{cases}$$
$$= \begin{cases} -(x^3-5x^2+4x) & if \quad x(x-1)(x-4) < 0 \\ x^3-5x^2+4x & if \quad x(x-1)(x-4) \geq 0 \end{cases}$$

To simplify the inequalities, we use the table method. Our division values are 0, 1 *and* 4.

x	$(-\infty,0)$	$\{0\}$	$(0,1)$	$\{1\}$	$(1,4)$	$\{4\}$	$(4,\infty)$
x	$-$	0	$+$	$+$	$+$	$+$	$+$
$(x-1)$	$-$	$-$	$-$	0	$+$	$+$	$+$
$(x-4)$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x(x-1)(x-4)$	$-$	0	$+$	0	$-$	0	$+$

So, our solution is:

$$|x^3-5x^2+4x| = \begin{cases} -(x^3-5x^2+4x) & \textit{if} \quad x<0 \\ x^3-5x^2+4x & \textit{if} \quad 0\le x\le 1 \\ -(x^3-5x^2+4x) & \textit{if} \quad 1<x<4 \\ x^3-5x^2+4x & \textit{if} \quad x\ge 4 \end{cases}$$

7.4 Exercises

Problems for Section 7.1

7.1 + 7.2 tonite 10/21

Problem 1. Solve. Graph your solutions on a number line.

(a) $3x+1\ge 2+x$ (b) $-1<2-\frac{x}{3}\le 1$ (c) $x^2-x-6>0$

(d) $(x+3)(x-2)^2(x-1)^4<0$ (e) $\frac{2x}{x-2}>0$ (f) $\frac{2}{x}\le\frac{x}{2}$

Problem 2. Solve. Write solutions in interval notation.

(a) $-1<\frac{3-x}{2}\le 1$ (b) $x^3+2x^2-4x-8\le 0$ (c) $\frac{1}{x}\ge\frac{1}{x+3}$

(d) $\frac{x}{2}-\frac{8x}{3}+\frac{x}{4}>\frac{23}{6}$ (e) $x^2-2x+1\le 0$

Problem 3. Solve. Write the solutions in algebraic notation.

(a) $-6x+3>x+5$ (b) $x^4-16<0$ (c) $\frac{2x}{x-2}>1$

(d) $\frac{-x^2+x}{x+2}>-x+3$ (e) $\frac{x+12}{x+2}-3\ge 0$ (f) $2^x(x-1)<0$

Problem 4. Find the domain for each of the following functions:

(a) $f(x)=\sqrt{x^2+4x+3}$ (b) $\sin^{-1}(1-x^2)$

Problem 5. For what non-negative integers n is it true that $\sum_{i=1}^{n} i<465$?

Problem 6. Your friend Cletus wrote: "$\frac{x}{3x+8}>\frac{x-2}{5}$, and so $5x>(x-2)(3x+8)$." Please explain to him (kindly) why this is not correct.

Problems for Section 7.2

Problem 1. Find numbers a and b to show that $|a+b|\ne|a|+|b|$.

Problem 2. Find numbers a and b to show that $|a-b|\ne|a|-|b|$.

Problem 3. For what values of x does $|3x-4| = 3x-4$?
For what values of x does $|3x-4| = -(3x-4)$?

Problem 4. Clive (Cletus' twin brother) wrote: "$|x-5| = x+5$." Explain to him (also kindly) why this is not correct.

Problem 5. Solve each of the following:

(a) $|2x| = x+1$ (b) $|-3x+6| = 9x$ (c) $|2x-5| = 9$

(d) $x - |x| = 1$ (e) $|x-10| = x^2 - 10x$ (f) $\left|\frac{x+3}{2x-1}\right| = 2$

(g) $|x^2+x| = |x-15|$ (h) $|1-2x| = 3 + |x+5|$ (i) $|x|^2 + |x| - 12 = 0$

Problem 6. Simplify:

(a) $\frac{|x+2|}{x+2}$ for $x \neq -2$ (b) $\frac{|x^2-4|}{|3x+6|}$ for $x \neq -2$

Problem 7. Express the following using absolute value:

(a) The distance between x and 3 is 12.
(b) x is four units away from 7.
(c) The distance between $2x$ and -4 is 1.
(d) x is six units from the origin.

Problem 8. Graph the following: (a) $y = |x-3| + 2$ (b) $y = |x^3 - 1|$ (c) $y = |-x|$
What does this last graph tell you about the function $f(x) = |x|$ concerning even/odd/neither?

Problems for Section 7.3

Problem 1. Express each of the following as an inequality statement involving absolute value:

(a) "x is less than 6 units from 4 on a number line."
(b) "y is at least 8 units from -1 on a number line."
(c) "z is no more than 5 units from the origin of a number line."
(d) $-3 \leq x \leq 3$ (e) $-4 < x < 10$
(f) $x < -4$ or $x > 4$ (g) $x < 2$ or $x > 10$

Problem 2. Solve for x. Express your answer in algebraic notation. For (a) and (b) also sketch your solution on a number line.

(a) $|x+7| < 5$ (b) $|1-2x| > 5$ (c) $\left|1 - \frac{2x}{3}\right| \leq 1$ (d) $\left|\frac{4-5x}{2}\right| \geq 1$

(e) $|\cos x - 243| \leq -2$ (f) $|x| > x+1$ (g) $3|x+2| - 5 > 10$ (h) $|x-1| > |3x-5|$

(i) $\frac{1}{|2x+7|} \leq \frac{1}{4}$ (j) $\left|\frac{x+3}{x-1}\right| \leq 3$ (k) $|x+3| + |x-2| > 6$ (Hint: Check four cases).

Problem 3. Rewrite as a piecewise function: $f(x) = |x^2 - 4x - 21|$ See Example 7.3.9.

Problem 4. Suppose $f(x) = x^2 - x$ and $g(x) = x^2 + 3x - 8$. For which values of x is the vertical distance between the graphs of f and g less than 4?

7.5 Answers to Exercises

Answers for Section 7.1 Exercises

Answer to Problem 1.

(a) $x \geq \frac{1}{2}$

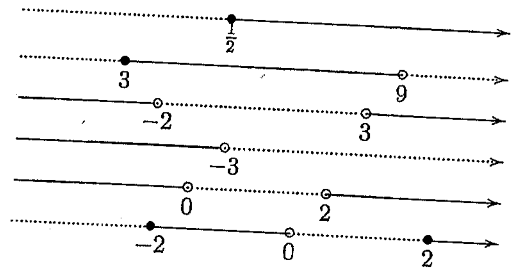
(b) $3 \leq x < 9$

(c) $x < -2$ or $x > 3$

(d) $x < -3$

(e) $x < 0$ or $x > 2$

(f) $-2 \leq x < 0$ or $x \geq 2$



Answer to Problem 2.

(a) $[1, 5)$ (b) $(-\infty, 2]$ (c) $(-\infty, -3) \cup (0, \infty)$ (d) $(-\infty, -2)$ (e) $[1]$

Answer to Problem 3.

(a) $x < -\frac{2}{7}$ (b) $-2 < x < 2$ (c) $x < -2$ or $x > 2$
(d) $x < -2$ (e) $-2 < x \leq 3$ (f) $x < 1$

Answer to Problem 4.

(a) $(-\infty, -3] \cup [-1, \infty)$ (b) $[-\sqrt{2}, \sqrt{2}]$

Answer to Problem 5.
$1 \leq n \leq 29$

Answer to Problem 6.
We don't know the sign of $(3x + 8)$ so we can't be sure which inequlity sign is valid.

Answers for Section 7.2 Exercises

Answer to Problem 1.
Answers will vary. opposite signed a + b

Answer to Problem 2.
Answers will vary.

Answer to Problem 3.
(a) $x \geq \frac{4}{3}$ (b) $x \leq \frac{4}{3}$.

Answer to Problem 4.
$$|x - 5| = \begin{cases} x - 5 & if \quad x \geq 5 \\ -x + 5 & if \quad x < 5 \end{cases}$$
$(x + 5)$ is not always non-negative.

Answer to Problem 5.

(a) $1, -\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $-2, 7$ (d) no solution (e) $-1, 10$
(f) $-\frac{1}{5}, \frac{5}{3}$ (g) $-5, 3$ (h) $-\frac{7}{3}, 9$ (i) $-3, 3$

Answer to Problem 6.

(a) $\dfrac{|x+2|}{x+2} = \begin{cases} -1 & if \quad x < -2 \\ 1 & if \quad x > -2 \end{cases}$ (b) $\dfrac{|x^2-4|}{|3x+6|} = \dfrac{|x-2|}{3}$ if $x \neq -2$

Answer to Problem 7.

(a) $|x-3| = 12$ (b) $|x-7| = 4$ (c) $|2x+4| = 1$ (d) $|x| = 6$

Answer to Problem 8.
Graphs not shown. (b) even

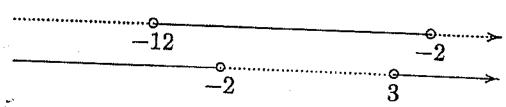
Answers for Section 7.3 Exercises

Answer to Problem 1.

(a) $|x-4| < 6$ (b) $|y+1| \geq 8$ (c) $|z| \leq 5$ (d) $|x| \leq 3$
(e) $|x-3| < 7$ (f) $|x| > 4$ (g) $|x-6| > 4$

Answer to Problem 2.

(a) $-12 < x < -2$
(b) $x < -2$ or $x > 3$



(c) $0 \leq x \leq 3$ (d) $x \leq \frac{2}{5}$ or $x \geq \frac{6}{5}$ (e) no solutions
(f) $x < -\frac{1}{2}$ (g) $x < -7$ or $x > 3$ (h) $\frac{3}{2} < x < 2$
(i) $x \leq -\frac{11}{2}$ or $x \geq -\frac{3}{2}$ (j) $x \leq 0$ or $x \geq 3$ (k) $x < -\frac{7}{2}$ or $x > \frac{5}{2}$

Answer to Problem 3.

$$|x^2-4x-21| = \begin{cases} x^2-4x-21 & if \quad x \leq -3 \\ -x^2+4x+21 & if \quad -3 < x < 7 \\ x^2-4x-21 & if \quad x \geq 7 \end{cases}$$

Answer to Problem 4.
$1 < x < 3$