Chapter 2

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Chapter 2

Miscellaneous Topics

This chapter contains four independent topics: Polynomial Long Division, Completing the Square, The Binomial Theorem, and Sigma Notation and Operations. These topics do not have to be covered at this time in the course. The section on Sigma Notation and Operations can be done at any time. The other three should be covered before the chapter "Polynomials."

2.1 Polynomial Long Division

In this section we review the algebra for dividing polynomials. It is very similar to the process for dividing integers. Let's look at an arithmetic problem first, and identify all of the steps involved.

We will rewrite $\dfrac{64,207}{23}$ by performing the division $2\,3\,\overline{)6\;4\;2\;0\;7}$

Working from the left, we decide that 23 will go into 64 twice, so 2 is the first digit in our quotient.

$$\begin{array}{rccccccl}
 & & 2 & & & & & \\
\cline{2-6}
2\,3\,) & 6 & 4 & 2 & 0 & 7 & & \\
 & 4 & 6 & & & & & \text{multiply 2 times 23 and put 46 here} \\
\cline{2-3}
 & 1 & 8 & 2 & & & & \text{subtract the 46 to get 18, and bring down the 2}
\end{array}$$

Now we decide that 23 goes into 182 seven times, so 7 is the next value in our quotient.

$$\begin{array}{rccccccl}
 & & 2 & 7 & & & & \\
\cline{2-6}
2\,3\,) & 6 & 4 & 2 & 0 & 7 & & \\
 & 4 & 6 & & & & & \\
\cline{2-3}
 & 1 & 8 & 2 & & & & \\
 & 1 & 6 & 1 & & & & \text{multiply 7 times 23 and put 161 here} \\
\cline{2-4}
 & & 2 & 1 & 0 & & & \text{subtract the 161 to get 21, and bring down the 0}
\end{array}$$

We repeat this process twice more to finish with:

$$
\begin{array}{rrrrrrl}
 & & 2 & 7 & 9 & 1 & \\
\cline{2-6}
23 \,\big) & 6 & 4 & 2 & 0 & 7 & \\
 & 4 & 6 & & & & \\
\cline{2-3}
 & 1 & 8 & 2 & & & \\
 & 1 & 6 & 1 & & & \\
\cline{2-4}
 & & 2 & 1 & 0 & & \\
 & & 2 & 0 & 7 & & \text{multiply 9 times 23 and put 207 here} \\
\cline{3-5}
 & & & & 3 & 7 & \text{subtract the 207 to get 3, and bring down the 7} \\
 & & & & 2 & 3 & \text{multiply 1 times 23 and put 23 here} \\
\cline{5-6}
 & & & & 1 & 4 & \text{subtract the 23 to get 14. This is the remainder.}
\end{array}
$$

We can interpret this result two ways:

$$\frac{64,207}{23} = 2,791\frac{14}{23} \quad \text{or,} \quad 64,207 = (23)(2,791) + 14.$$

Keep this process of arithmetic long division in mind as you look at the examples below for polynomial long division. The methods are the same.

Example 2.1.1.

$$x - 3\,\big)\overline{\;4x^4 \;-4x^3 - 25x^2 \;+x + 6}$$

Check to make sure that the divisor and dividend are both written so that their terms are in order of declining powers of the variable. We are OK here.

Now look at the first term in the dividend and the first term in the divisor. What term do we need to multiply times the first term in the divisor to get exactly the first term in the dividend? We need $4x^3$. This then is our first term in the quotient.

$$
\begin{array}{rl}
 & \;4x^3 \\
\cline{2-2}
x - 3\,\big) & 4x^4 \;-4x^3 - 25x^2 \;+x + 6
\end{array}
$$

Now we multiply the quotient term $4x^3$ times each term in the divisor, getting $4x^4 - 12x^3$ and place that expression in the appropriate place under the dividend. We will want to subtract this entire expression from the dividend. The subtraction step is one where many errors are made. It can be a good idea to negate (change the sign of) all of the terms in the expression beneath the dividend and then add them instead of subtract. This is done here.

$$
\begin{array}{rl}
 & \;4x^3 \\
\cline{2-2}
x - 3\,\big) & 4x^4 \;-4x^3 - 25x^2 \;+x + 6 \\
 & \underline{-\,4x^4 + 12x^3}
\end{array}
$$

We perform the subtraction (here adding the negation) and then bring down the next term.

$$
\begin{array}{rl}
 & \;4x^3 \\
\cline{2-2}
x - 3\,\big) & 4x^4 \;-4x^3 - 25x^2 \;+x + 6 \\
 & \underline{-\,4x^4 + 12x^3} \\
 & \qquad\;\; 8x^3 - 25x^2
\end{array}
$$

We now look at the first term in the difference expression, $(8x^3 - 25x^2)$, and the first term in the divisor. What term do we need to multiply times x to get exactly $8x^3$? The answer, $8x^2$, becomes the second term in the quotient.

$$
\begin{array}{rrrrrr}
 & & 4x^3 & +8x^2 & & \\
\cline{2-6}
x-3) & 4x^4 & -4x^3 & -25x^2 & +x & +6 \\
 & -4x^4 & +12x^3 & & & \\
\cline{2-3}
 & & 8x^3 & -25x^2 & & \\
\end{array}
$$

As before, we multiply the quotient term $8x^2$ times each term in the divisor, place the new terms under the previous difference expression, and then negate the terms as the first part of the subtraction process.

$$
\begin{array}{rrrrrr}
 & & 4x^3 & +8x^2 & & \\
\cline{2-6}
x-3) & 4x^4 & -4x^3 & -25x^2 & +x & +6 \\
 & -4x^4 & +12x^3 & & & \\
\cline{2-3}
 & & 8x^3 & -25x^2 & & \\
 & & -8x^3 & +24x^2 & & \\
\cline{3-4}
\end{array}
$$

We add (to complete the subtraction process) and bring down the next term.

$$
\begin{array}{rrrrrr}
 & & 4x^3 & +8x^2 & & \\
\cline{2-6}
x-3) & 4x^4 & -4x^3 & -25x^2 & +x & +6 \\
 & -4x^4 & +12x^3 & & & \\
\cline{2-3}
 & & 8x^3 & -25x^2 & & \\
 & & -8x^3 & +24x^2 & & \\
\cline{3-4}
 & & & -x^2 & +x & \\
\end{array}
$$

We now repeat the division step, getting $-x$ as our next quotient term.

$$
\begin{array}{rrrrrr}
 & & 4x^3 & +8x^2 & -x & \\
\cline{2-6}
x-3) & 4x^4 & -4x^3 & -25x^2 & +x & +6 \\
 & -4x^4 & +12x^3 & & & \\
\cline{2-3}
 & & 8x^3 & -25x^2 & & \\
 & & -8x^3 & +24x^2 & & \\
\cline{3-4}
 & & & -x^2 & +x & \\
\end{array}
$$

We repeat the multiplication and negation step.

$$
\begin{array}{rrrrrr}
 & & 4x^3 & +8x^2 & -x & \\
\cline{2-6}
x-3) & 4x^4 & -4x^3 & -25x^2 & +x & +6 \\
 & -4x^4 & +12x^3 & & & \\
\cline{2-3}
 & & 8x^3 & -25x^2 & & \\
 & & -8x^3 & +24x^2 & & \\
\cline{3-4}
 & & & -x^2 & +x & \\
 & & & x^2 & -3x & \\
\cline{4-5}
\end{array}
$$

Then we add and bring down the next term.

$$
\begin{array}{r|l}
 & 4x^3 + 8x^2 - x \\
\hline
x-3 & 4x^4 - 4x^3 - 25x^2 + x + 6 \\
 & -4x^4 + 12x^3 \\
\hline
 & 8x^3 - 25x^2 \\
 & -8x^3 + 24x^2 \\
\hline
 & -x^2 + x \\
 & x^2 - 3x \\
\hline
 & -2x + 6
\end{array}
$$

One more division, multiplication, negation ...

$$
\begin{array}{r|l}
 & 4x^3 + 8x^2 - x - 2 \\
\hline
x-3 & 4x^4 - 4x^3 - 25x^2 + x + 6 \\
 & -4x^4 + 12x^3 \\
\hline
 & 8x^3 - 25x^2 \\
 & -8x^3 + 24x^2 \\
\hline
 & -x^2 + x \\
 & x^2 - 3x \\
\hline
 & -2x + 6
\end{array}
$$

One more subtraction, and ...

$$
\begin{array}{r|l}
 & 4x^3 + 8x^2 - x - 2 \\
\hline
x-3 & 4x^4 - 4x^3 - 25x^2 + x + 6 \\
 & -4x^4 + 12x^3 \\
\hline
 & 8x^3 - 25x^2 \\
 & -8x^3 + 24x^2 \\
\hline
 & -x^2 + x \\
 & x^2 - 3x \\
\hline
 & -2x + 6 \\
 & 2x - 6 \\
\hline
 & 0
\end{array}
$$

... we are finished. There is no remainder this time. We can interpret our result two ways:

$$\frac{4x^4 - 4x^3 - 25x^2 + x + 6}{x-3} = 4x^3 + 8x^2 - x - 2 \textit{ when } x \neq 3$$

$$\textit{or} \quad 4x^4 - 4x^3 - 25x^2 + x + 6 = (x-3)(4x^3 + 8x^2 - x - 2)$$

When doing polynomial long division it is easy to make a mistake in the subtraction step. It should be stressed again that you need to be sure to correctly subtract the *entire* expression by distributing the negative sign through the entire expression before adding. If you find that you are getting incorrect answers (this is a hypothetical situation of course) check the subtraction steps.

In the example above the dividend contained a term for each power of x from 4 to 0 (the constant term). In Example 2.1.2 the dividend is $x^3 + x - 5$. There is no written x-squared term. When you do long division it is a good idea to include "missing" terms, using zero as the coefficient. We can explicity write "$+0x^2$" or we can just leave a blank space where the missing term should go.

Example 2.1.2. *Divide:* $\dfrac{x^3+x-5}{x+3}$

$$
\begin{array}{rrrrr}
 & x^2 & -3x & +10 & \\
\cline{2-5}
x+3 \big) & x^3 & & +x & -5 \\
 & -x^3 & -3x^2 & & \\
\cline{2-3}
 & & -3x^2 & +x & \\
 & & 3x^2 & +9x & \\
\cline{3-4}
 & & & 10x & -5 \\
 & & & -10x & -30 \\
\cline{4-5}
 & & & & -35
\end{array}
$$

So, $\dfrac{x^3+x-5}{x+3} = x^2 - 3x + 10 + \dfrac{-35}{x+3}$

or $x^3 + x - 5 = (x^2 - 3x + 10)(x+3) - 35$

In Example 2.1.2 the division did not "come out evenly." We can stop dividing as soon as the power of the divisor is greater than the power of a difference expression. The rest is the remainder. There are two ways to think of this result, as written in the example.

Here we have one more example. This time the divisor is "missing" a term. Observe how the terms must be aligned.

Example 2.1.3. *Divide:* $(5x^4 - 4x^3 + x^2 - 7x + 2) \div (2x^2 + 1)$

$$
\begin{array}{rrrrrr}
 & & & \frac{5}{2}x^2 & -2x & -\frac{3}{4} \\
\cline{2-6}
2x^2+1 \big) & 5x^4 & -4x^3 & +x^2 & -7x & +2 \\
 & -5x^4 & & -\frac{5}{2}x^2 & & \\
\cline{2-4}
 & & -4x^3 & -\frac{3}{2}x^2 & -7x & \\
 & & 4x^3 & & +2x & \\
\cline{3-5}
 & & & -\frac{3}{2}x^2 & -5x & +2 \\
 & & & \frac{3}{2}x^2 & & +\frac{3}{4} \\
\cline{4-6}
 & & & & -5x & +\frac{11}{4}
\end{array}
$$

So, $\dfrac{5x^4 - 4x^3 + x^2 - 7x + 2}{2x^2+1} = \frac{5}{2}x^2 - 2x - \frac{3}{4} + \dfrac{-5x + \frac{11}{4}}{2x^2+1}$

or $5x^4 - 4x^3 + x^2 - 7x + 2 = (2x^2+1)(\frac{5}{2}x^2 - 2x - \frac{3}{4}) + (-5x + \frac{11}{4})$

2.2 Completing the Square

In this section we deal with expressions in the form $ax^2 + bx + c$ where $a, b,$ and c are in $\Re$. We will rewrite them so that $ax^2 + bx + c = a(x-h)^2 + k$ where h and k are also in $\Re$.

In Chapter 1 we learned to recognize perfect squares for factoring.

Perfect Square Equation: $x^2 + 2cx + c^2 = (x + c)^2$

Study the equation above. Look at the pattern. On the left, the coefficient of the x term ($2c$) is twice the value of some constant c whose square (c^2) is the final term of that expression. On the right, that same c is part of a squared sum.

Suppose we have $x^2 + 6x$. What constant could we add to this expression so that it would be a perfect square expression? The coefficient of the x term is 6. Six is twice three. So $3^2 = 9$ is the constant we need. $x^2 + 6x + 9 = (x + 3)^2$.

Oooh, that was fun. Let's try another one. Suppose we have $x^2 - 8x$. What constant would we need to add so that we create a perfect square? Half of -8 is -4. $(-4)^2 = 16$, so we add 16. $x^2 - 8x + 16 = (x - 4)^2$.

Look back at the Perfect Square Equation. What is the value of c that we found in each of the two examples?

Suppose we have $x^2 + 5x$. We still work the same way. This time $c = \frac{5}{2}$.
So, $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$ is the constant we need. $x^2 + 5x + \frac{25}{4} = (x + \frac{5}{2})^2$.

Be careful. We are <u>not</u> saying that $x^2 + 6x = (x + 3)^2$ or that $x^2 - 8x = (x - 4)^2$ or that $x^2 + 5x = (x + \frac{5}{2})^2$. These statements are all false. In every instance above we added a constant. In order to maintain an *equation* we would have to subtract off the same constant. If we wanted to write statements equivalent to the originals, they would look like:

$$x^2 + 6x = x^2 + 6x + 9 - 9 = (x^2 + 6x + 9) - 9 = (x + 3)^2 - 9$$
$$x^2 - 8x = x^2 - 8x + 16 - 16 = (x^2 - 8x + 16) - 16 = (x - 4)^2 - 16$$
$$x^2 + 5x = x^2 + 5x + \tfrac{25}{4} - \tfrac{25}{4} = (x^2 + 5x + \tfrac{25}{4}) - \tfrac{25}{4} = (x + \tfrac{5}{2})^2 - \tfrac{25}{4}$$

Each of the above expressions could be written as the sum of a perfect square and a constant. The process of finding the square term is referred to as "Completing the Square." Let's look at some variations. First we will do two examples where a constant term is in the original expression.

Example 2.2.1.
Write $x^2 + 10x - 1$ as the sum of a perfect square and a constant.

$$\begin{aligned} x^2 + 10x - 1 &= (x^2 + 10x) - 1 \\ &= (x^2 + 10x + 25) - 25 - 1 \\ &= (x + 5)^2 - 26 \end{aligned}$$

Example 2.2.2.
Write $x^2 - 3x + 4$ as the sum of a perfect square and a constant.

$$\begin{aligned} x^2 - 3x + 4 &= (x^2 - 3x) + 4 \\ &= \left(x^2 - 3x + \frac{9}{4}\right) - \frac{9}{4} + 4 \\ &= \left(x - \frac{3}{2}\right)^2 + \frac{7}{4} \end{aligned}$$

We will not always have the pleasure of having a 1 as the coefficient of the x^2 term. We can still complete the square. We do this by first factoring the coefficient out of the x^2 and the x terms. Then we complete the square as done above, except that we have to be very careful when we add our compensating constant. Look at the following example.

Example 2.2.3.
Write $2x^2 - 12x + 1$ *as the sum of a perfect square and a constant.*

$$\begin{aligned} 2x^2 - 12x + 1 &= 2(x^2 - 6x) + 1 \\ &= 2(x^2 - 6x + 9) - 18 + 1 \\ &= 2(x - 3)^2 - 17 \end{aligned}$$

Why did we add -18 in the second line instead of -9? We needed the 9 to complete the perfect square. But that 9 was placed inside parentheses, and everything in that parentheses gets multiplied by the outside 2. So, we were really adding $2 \cdot 9 = 18$ to our original expression. So, we needed to subtract 18 to compensate.

Example 2.2.4.
Write $-x^2 - 2x + 6$ *as the sum of a perfect square and a constant.*

$$\begin{aligned} -x^2 - 2x + 6 &= -(x^2 + 2x) + 6 \\ &= -(x^2 + 2x + 1) + 1 + 6 \\ &= -(x + 1)^2 + 7 \end{aligned}$$

This time we had to add 1 to keep our equation balanced. Do you see why?

Comprehension Check 2.1.

Write $3x^2 + 9x - 2$ *as the sum of a perfect square and a constant. Justify each step. Your result should be* $3x^2 + 9x - 2 = 3\left(x + \frac{3}{2}\right)^2 - \frac{35}{4}$.

At the beginning of this section we said that we would rewrite expressions of the form $ax^2 + bx + c$ into expressions in the form $a(x - h)^2 + k$. We have done that. This last expression is simply the sum of a perfect square and a constant. In Example 2.2.3 $a = 2, b = -12, c = 1, h = 3$ and $k = -17$. Notice that the coefficient for the x^2 term, the a, becomes the coefficient for the perfect square sum. What are the a, b, c, h, k values for Example 2.2.1, Example 2.2.4 and the problem in Comprehension Check 2.1?

2.3 The Binomial Theorem

In this section we develop an easy way for expanding expressions in the form $(a + b)^n$ where n is a non-negative integer. First, we need some notation.

2.3.1 Factorials

In mathematics the exclamation point (!) is used to indicate a factorial. To understand what the factorial of a number is, consider the following example.

Example 2.3.1.
$3! = 3 \cdot 2 \cdot 1 = 6$
$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$
$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3,628,800$

What is 4!? (yes, work it out). Did you get 24? Good. How about 1!? (you don't need the trusty pencil for this one) $1! = 1$.

Factorials are defined for non-integers (you will meet them in calculus) but they are not needed here so we will restrict our discussion to factorials of non-negative integers.

Zero is a non-negative integer. By definition, $0! = 1$ (yes, "one", this is not an occassion to alert your instructor to a typo). We can write the following definition:

Definition 2.3.1.
For n a positive integer, we define "n factorial", written n!, to be:

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

and, further, we define $0! = 1$.

Since factorials come already factored it is easy to work with fractions that involve products of factorials in the numerators and denominators.
We saw above that $5! = 120$ and $3! = 6$, so $\frac{5!}{3!} = \frac{120}{6} = 20$.

We could save ourselves the time of calculating the 120 and 6 and evaluate $\frac{5!}{3!}$ from the factorial definition.
$\frac{5!}{3!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = \frac{5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = \frac{5 \cdot 4}{1} = 20$

We can even shortcut our writing by using the factorial notation for the parts we know will cancel out:
$\frac{5!}{3!} = \frac{5 \cdot 4 \cdot \cancel{3}!}{\cancel{3}!} = 5 \cdot 4 = 20$

Example 2.3.2.

$$\frac{9!}{6!} = \frac{9 \cdot 8 \cdot 7 \cdot \cancel{6}!}{\cancel{6}!} = 9 \cdot 8 \cdot 7 = 504$$

$$\frac{9!}{3!\,6!} = \frac{9 \cdot 8 \cdot 7 \cdot \cancel{6}!}{3 \cdot 2 \cdot \cancel{6}!} = \frac{{}^{3}\cancel{9} \cdot {}^{4}\cancel{8} \cdot 7}{\cancel{3} \cdot \cancel{2}} = 12 \cdot 7 = 84$$

Combinations

One branch of mathematics where factorials are used extensively is Probability. In probability we often have to count how many subsets there are of a particular set. This counting is called finding "combinations." We will not go into the probability application here, but we will use the notation and verbage.

Definition 2.3.2. *Combinations of n things taken r at a time or simply "n choose r" for short, is denoted ${}_nC_r$ or $\binom{n}{r}$ and is defined to be:*

$$ {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} \text{ for non-negative integers, } n \text{ and } r, \text{ where } n \geq r. $$

We will use the $\binom{n}{r}$ notation almost exclusively, but the ${}_nC_r$ is very common also so you should be aware of it.

Example 2.3.3.

$\binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7!}{3!\,4!} = \frac{7\cdot 6\cdot 5\cdot 4!}{3\cdot 2\cdot 4!} = \frac{7\cdot \cancel{6}\cdot 5\cdot \cancel{4!}}{\cancel{3}\cdot \cancel{2}\cdot \cancel{4!}} = 35$

$\binom{9}{3} = 84$. *This was calculuated in the second part of Example 2.3.2.*

Let's compare $\binom{12}{2}$ and $\binom{12}{10}$.

$\binom{12}{2} = \frac{12!}{2!\,10!}$ and $\binom{12}{10} = \frac{12!}{10!\,2!}$. We don't have to finish the calculations to see that they are the same.

$\binom{8}{5} = \frac{8!}{5!\,3!} = \frac{8!}{3!\,5!} = \binom{8}{3}$.

What value of x (besides 9) makes this true: $\binom{15}{9} = \binom{15}{x}$?

We can generalize our discovery by writing: $\binom{n}{r} = \binom{n}{n-r}$.

In our definition of $\binom{n}{r}$ we said that $n \geq r$. If $n = r$ we get $\binom{n}{n} = \frac{n!}{n!\,0!}$. Remember that 0! is defined to be 1. So, $\binom{n}{n} = \frac{n!}{n!\cdot 1} = 1$.

$\binom{n}{0}$ is also 1. (Why?)

In the following Important Idea we list our findings so far, and include two more. Think about these items, or play with some numbers as examples, until you are convinced that they are true.

Important Idea 2.3.1.

1. $\binom{n}{r} = \binom{n}{n-r}$
2. $\binom{n}{n} = 1$

3. $\binom{n}{0} = 1$

4. $\binom{n}{1} = n$

5. $\binom{n}{r}$, *as defined above, will always be a positive integer.*

Pascal's Triangle

There is a shortcut way to find the values for $\binom{n}{r}$ combinations without having to calculate factorials. This method works for all values of n and r but it ceases to be a shortcut when n is big, say greater than 8. This shortcut is Pascal's Triangle.[1]

Below is the beginning (top) of Pascal's Triangle.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

The triangle is made of rows of numbers. The top row contains only one number. Each subsequent row contains one more number than the preceding row. Thus, we build a triangle shape, much like the arrangement of bowling pins.

Each row starts and ends with a 1.

Each number that is not a 1 is gotten by adding the two numbers in the preceding row which are immediately to the left and right of the position of that number. For example, the 2 in the third row is gotten by adding the 1 and the 1 in the second row. The 6 in the fifth row is gotten by adding the two 3's in the fourth row. The first 15 in the seventh row is gotten by adding the first 5 and the first 10 in the sixth row. We can continue to lengthen the triangle. The next row will have numbers $1, 7, 21, 35, 35, 21, 7, 1$. Put these values on the triangle. What will be in the row after that?

The values in Pascal's Triangle correspond to the values for Combinations.

[1]Blaise Pascal (1623-1662) was a French mathematician who didn't have a laptop.

$$\binom{0}{0}$$
$$\binom{1}{0}\binom{1}{1}$$
$$\binom{2}{0}\binom{2}{1}\binom{2}{2}$$
$$\binom{3}{0}\binom{3}{1}\binom{3}{2}\binom{3}{3}$$
$$\binom{4}{0}\binom{4}{1}\binom{4}{2}\binom{4}{3}\binom{4}{4}$$
$$\binom{5}{0}\binom{5}{1}\binom{5}{2}\binom{5}{3}\binom{5}{4}\binom{5}{5}$$

Look at both Pascal's triangle and the triangle of Combinations above. Notice that the $\binom{n}{0}$ and $\binom{n}{n}$ terms are indeed all 1's.

Look at the row symmetry in Pascal's triangle. This is consistent with the idea that $\binom{n}{r} = \binom{n}{n-r}$.

In Example 2.3.3 we saw that $\binom{7}{3} = 35$. You can see that 35 in the row that you added to the triangle. It is the first 35. Remember that when you start counting, the first row is $\binom{0}{0}$ and the first term in each row is the $\binom{n}{0}$ term, so the $\binom{7}{3}$ is the fourth term in the eighth row, not the third term in the seventh row.

Comprehension Check 2.2.

1. *Find the value for* $\binom{5}{3}$ *in Pascal's triangle. Then calculate this value using the factorial definition.*
2. *Add yet another row to your Pascal's Triangle.*

We will come back to Combinations and Pascal shortly.

Binomial Expansions

You recall that $(a+b)^2 = a^2 + 2ab + b^2$.

We can expand $(a+b)^3$ to get $(a+b)^3 = (a+b)(a+b)^2 =$
$(a+b)(a^2+2ab+b^2) = a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

We can similarly use the distributive property repeatedly to get expansions for $(a+b)^n$ for any non-negative integer n. We list the results of these expansions below, for $n \leq 5$.

$$\begin{aligned}
&(a+b)^0 = 1\\
&(a+b)^1 = a+b\\
&(a+b)^2 = a^2+2ab+b^2\\
&(a+b)^3 = a^3+3a^2b+3ab^2+b^3\\
&(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4\\
&(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5
\end{aligned}$$

Look at the patterns evolving in these expansions. In each expansion,

- what happens to the powers of a? What is the highest power of a?
- what happens to the powers of b? What is the highest power of b?
- what is the sum of the powers of a and b in each term?
- what is the sequence of coefficients for the terms? Note the symmetry.

Can you use your observations to find the expansion for $(a+b)^6$?

- The powers of a will begin at 6 and decrease to 0.
- The powers of b will begin at 0 and increase to 6.
- The coefficients for the terms are the combinations $\binom{6}{0}, \binom{6}{1}, \ldots, \binom{6}{6}$, whose values we can get from Pascal's triangle.

So, we have $(a+b)^6 = a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$.

The Binomial Theorem

Theorem 2.3.1. *The Binomial Theorem*

$$\begin{aligned}
(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots\\
\ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n
\end{aligned}$$

where $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$

Example 2.3.4.
Expand $(a+b)^7$.

$$\begin{aligned}
(a+b)^7 &= \binom{7}{0}a^7 + \binom{7}{1}a^6b + \binom{7}{2}a^5b^2 + \binom{7}{3}a^4b^3 + \binom{7}{4}a^3b^4\\
&\quad + \binom{7}{5}a^2b^5 + \binom{7}{6}ab^6 + \binom{7}{7}b^7\\
&= a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7
\end{aligned}$$

If we again look at the pattern for the binomial expansion we see that $\binom{n}{r}$ and $\binom{n}{n-r}$, which are of course equal, are the coefficients for the $a^r b^{n-r}$ and $a^{n-r}b^r$ terms. And, as noted before, the exponents on a and b for any term will always sum to n. Observing these symmetries gives some insight into the algebra of expanding $(a+b)^n$ and can help you avoid errors.

Example 2.3.5.
What is the coefficient of x^3y^5 in the expansion of $(x+y)^8$?

The power of the expansion is 8, so $n = 8$. The exponents of x and y are 3 and 5. So the coefficient for the x^3y^5 term is $\binom{8}{5}$, or its equivalent $\binom{8}{3}$. $\binom{8}{5} = \frac{8!}{5!\,3!} = \frac{8\cdot 7\cdot 6\cdot 5!}{5!\cdot 3\cdot 2} = \frac{8\cdot 7\cdot \cancel{6}\cdot \cancel{5}!}{\cancel{5}!\cdot \cancel{3}\cdot \cancel{2}} = 56.$

Example 2.3.6.

1. *What is the third term in the expansion $(x+y)^{12}$?*

   In the expansion, the powers of x decrease, starting at 12, so the third term will have x^{10}. Thus the y factor is y^2 and the coefficient is $\binom{12}{2}$. We calculate $\binom{12}{2} = \frac{12\cdot 11\cdot 10!}{10!\cdot 2} = 66$. So the third term in the expansion of $(x+y)^{12}$ is $66\,x^{10}y^2$.

2. *What other term has coefficient 66?*

   By symmetry, the other term is $66\,x^2y^{10}$.

Now we make it a bit more interesting. We will have other values for a and b. The process is exactly the same.

Example 2.3.7.
Expand $(a+2)^4$.

$$\begin{aligned}(a+2)^4 &= \binom{4}{0}a^4 + \binom{4}{1}a^3(2) + \binom{4}{2}a^2(2)^2 + \binom{4}{3}a(2)^3 + \binom{4}{4}(2)^4\\ &= a^4 + 4a^3\cdot 2 + 6a^2\cdot 4 + 4a\cdot 8 + 16\\ &= a^4 + 8a^3 + 24a^2 + 32a + 16\end{aligned}$$

Of course we do not have obvious symmetry of the coefficients in the final answer here.

In the next two examples we skip the combinations notation and go straight to the corresponding values found in Pascal's triangle. Example 2.3.8 shows that when we need to expand a differnce of terms, we can rewrite that difference as a sum.

Example 2.3.8.
Expand $(x-1)^5$.

$$\begin{aligned}(x-1)^5 &= [x+(-1)]^5\\ &= x^5 + 5x^4(-1) + 10x^3(-1)^2 + 10x^2(-1)^3 + 5x(-1)^4 + (-1)^5\\ &= x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1\end{aligned}$$

What pattern do you see with the signs in the final answer of the previous example? Do you think that signs will always alternate when expanding a difference?

Example 2.3.9 reminds us to watch the base carefully when using exponents.

Example 2.3.9.
Expand $(2a+b)^3$

$$\begin{aligned}(2a+b)^3 &= [(2a)+b]^3 \\ &= (2a)^3 + 3(2a)^2 b + 3(2a)b^2 + b^3 \\ &= 8a^3 + 3\cdot 4a^2 b + 3\cdot 2ab^2 + b^3 \\ &= 8a^3 + 12a^2 b + 6ab^2 + b^3\end{aligned}$$

Example 2.3.10.
Find the sixth term in the expansion $(3x+4y)^9$.

The powers for $(3x)$ *decrease, starting at* 9, *so for the sixth term the* $(3x)$ *factor will be* $(3x)^4$. *This tells us that the power on the* $(4y)$ *term must be* $9-4=5$. *The coefficient is* $\begin{pmatrix} 9 \\ 5 \end{pmatrix} = \frac{9!}{5!\,4!} = \frac{9\cdot 8\cdot 7\cdot 6\cdot 5!}{5!\cdot 4\cdot 3\cdot 2} = \frac{{}^3\cancel{9}\cdot \cancel{8}\cdot 7\cdot 6\cdot \cancel{5!}}{\cancel{5!}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}} = 126$. *So the sixth term in the expansion is* $(126)(3x)^4(4y)^5 = 126\cdot 3^4\cdot 4^5 x^4 y^5 = 10{,}450{,}944\,x^4y^5$

Certainly this last simplification is easier when done with a calculator.

Example 2.3.11.
Expand: $(x^{\frac{3}{5}}+3)^4$

$$\begin{aligned}(x^{\frac{3}{5}}+3)^4 &= (x^{\frac{3}{5}})^4 + 4(x^{\frac{3}{5}})^3\cdot 3 + 6(x^{\frac{3}{5}})^2\cdot 3^2 + 4(x^{\frac{3}{5}})\cdot 3^3 + 3^4 \\ &= x^{\frac{12}{5}} + 12x^{\frac{9}{5}} + 54x^{\frac{6}{5}} + 108x^{\frac{3}{5}} + 81\end{aligned}$$

2.4 Sigma (Σ) Notation and Operations

In this section we look at a notation used to represent lengthy sums and some ways we manipulate the sums.

2.4.1 Basic Notation

The symbol Σ (the Greek letter Sigma) is used to indicate the *sum* of numbers. The form is:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_{n-1} + a_n,$$

where m and n represent integers, $m \le n$, and the a_i terms represent real numbers. The "i" is called the *index of summation.* It takes on the integer values between m and n, inclusive.

This might look a little complicated the first time you see it. Think about the integer i taking on the values of m through n, one at a time, in succession. As i takes on each value, a real number called a_i is associated with it. Then, all of the a_i numbers are added. The final result is the Sigma value.

In Example 2.4.1 we have $m = 2$, $n = 7$ and for each i (between 2 and 7, inclusive) we simply have $a_i = i$. Then we add the a_i values for the final result.

Example 2.4.1.

$$\sum_{i=2}^{7} i = 2 + 3 + 4 + 5 + 6 + 7 = 27$$

In Example 2.4.2 we have $m = 0$, $n = 3$ and for each i (between 0 and 3, inclusive) we have $a_i = (i^2 - 1)$.

Example 2.4.2.

$$\sum_{i=0}^{3} (i^2 - 1) = (0^2 - 1) + (1^2 - 1) + (2^2 - 1) + (3^2 - 1) = 10$$

What are the values for m, n, and a_i in the next example?

Example 2.4.3.

$$\sum_{i=1}^{3} \frac{i}{i+1} = \frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} = \frac{23}{12}$$

Example 2.4.4.

$$\begin{aligned} \sum_{i=3}^{7} (-1)^i (2i) &= (-1)^3(2 \cdot 3) + (-1)^4(2 \cdot 4) + (-1)^5(2 \cdot 5) + (-1)^6(2 \cdot 6) + (-1)^7(2 \cdot 7) \\ &= -6 + 8 - 10 + 12 - 14 = -10. \end{aligned}$$

In Example 2.4.4 notice the use of $(-1)^i$ to obtain the alternating signs. How do you suppose you would write the sigma expression if you wanted to change the signs to get $6 - 8 + 10 - 12 + 14$?

Example 2.4.5. *(If you have studied the Trigonometry chapter)*

$$\begin{aligned} \sum_{i=2}^{5} cos(\pi i) i &= cos(2\pi) \cdot 2 + cos(3\pi) \cdot 3 + cos(4\pi) \cdot 4 + cos(5\pi) \cdot 5 \\ &= (1) \cdot 2 + (-1) \cdot 3 + (1) \cdot 4 + (-1) \cdot 5 = 2 - 3 + 4 - 5 = -2. \end{aligned}$$

Using cosine values is another way to generate an alternating sign. However, this method is obnoxious.

Comprehension Check 2.3.

1. *Expand the following sum and simplify:* $\sum_{i=1}^{6} (i^2 - i)$

2. *Write the sum $10 + 9 + 8 + 7 + 6$ in sigma notation. Compare your result with that of a classmate.*

If you compare your answer in Part 2 of Comprehension Check 2.3 with the answers of a few classmates you will likely find that there are several correct ways to write a sum using sigma notation. Probably there was variation in the choices for m and n, the lower and upper bounds of the index of summation.

Changing Bounds on the Index of Summation

A sum can be written in more than one way, by changing the bounds on the index of summation and making the corresponding change in the form of each term.

Example 2.4.6.

The sum in Example 2.4.3 could have been written

$$\sum_{i=0}^{2} \frac{i+1}{i+2} \quad \textit{or as} \quad \sum_{i=2}^{4} \frac{i-1}{i} \quad \textit{or even as} \quad \sum_{i=23}^{25} \frac{(i-22)}{(i-21)}.$$

Verify these sums by expanding them. Look at the pattern. Notice that when we increase (or decrease) the lower bound on the index of summation, m we do the same thing with the upper bound n, thus always keeping the same number of terms in our sum. Each of the sums above still contains three terms. Also, when we increase (or decrease) the bounds on the index of summation we have to decrease (or increase) the corresponding a_i expression. We do this by changing the value of i down (or up) to compensate for the change in the m and n. What would the summation look like if you started with lower bound $m = 7$?

When writing an equivalent sum by changing the bounds for the index of summation you should always check your new sum to make sure that its terms match those of original sum. With a long sum (more than five terms) it is probably sufficient to just check the first two and then the last term in the new sum.

Example 2.4.7.

Change the lower index of summation of the sum in Example 2.4.4 from a 3 to a 0.

We want to write a sum that begins with $i = 0$ and that is equivalent to $\sum_{i=3}^{7}(-1)^i(2i)$. We are decreasing the lower bound of the index of summation by three, so the the upper bound must be decreased from 7 to 4. Our new sigma expression will then have five terms, which is consistent with the original. Next we have to change the a_i terms. Since we decreased the value of i by three, we need to increase by three the input to the a_i terms to compensate. So, we will replace each i in the a_i term with $(i + 3)$. This gives us a resulting sum of $\sum_{i=0}^{4}(-1)^{(i+3)}(2(i+3))$. Check the terms of this new sum to make sure that they match those of the original sum.

We were careful to use parentheses when we wrote our new sum in Example 2.4.7. Without the parentheses, our sum would have been: $\sum_{i=0}^{4}(-1)^{i+3}(2i+3)$. How would this have been different? Be sure that when you make the adjustment in the i's of the a_i terms that you use any necessary parentheses.

We can generalize this idea with the following Important Idea.

Some Special Sums

Below are some "Special Sums". Notice that the following sums all have bottom index $m = 1$.

1. $\sum_{i=1}^{n} c = nc$ (established in Example 2.4.8)

2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

Sums 3 and 4 are offered without explanation. Sum 2 is quite useful for many applications. It is more appreciated when you see why it works, so we take a few minutes here to explain.

Consider $\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$.

If n is an even number, we rewrite the sum (you recall that we can add numbers in any order) above by pairing the terms that add up to $(n+1)$. That is, we match the first and last terms, then the second and second-last terms, and so forth. This is written:

$$\begin{aligned} &1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \\ &\quad = [1+n] + [2+(n-1)] + [3+(n-2)] + \cdots + \left[\tfrac{n}{2} + \left(\tfrac{n}{2}+1\right)\right]. \end{aligned}$$

There are $\frac{n}{2}$ sums of $(n+1)$, for a total, then, of $\frac{n(n+1)}{2}$.

If n is an odd number we rewrite $\sum_{i=1}^{n} i = \left(\sum_{i=1}^{n-1} i\right) + n$. Since $(n-1)$ must be even, we follow the same procedure as above, getting $\sum_{i=1}^{n-1} i$ equal to $\frac{(n-1)}{2}$ sums of n, for a total of $\frac{(n-1)n}{2}$. We then add the final n. Algebraically expressed all of this is:

$$\sum_{i=1}^{n} i = \left(\sum_{i=1}^{n-1} i\right) + n = \frac{(n-1)n}{2} + n = \frac{n^2 - n + 2n}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2}.$$

So, regardless of whether n was even or odd we got the result that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

We can use these "Special Sums" and the rules from Important Idea 2.4.2 to evaluate more complex sums:

Example 2.4.11.

$\sum_{i=1}^{100} 40 = 40 \cdot 100 = 4,000.$

Example 2.4.12.

$$\sum_{i=1}^{12} i^2 = \frac{12(12+1)(2\cdot 12+1)}{6} = \frac{12\cdot 13\cdot 25}{6} = 650.$$

Example 2.4.13.

$$51+52+53+\cdots+73 = \sum_{i=1}^{73} i - \sum_{i=1}^{50} i = \frac{(73)(74)}{2} - \frac{(50)(51)}{2} = 1,426.$$

Example 2.4.13 – Alternate Solution

$$51+52+53+\cdots+73 = \sum_{i=51}^{73} i = \sum_{i=1}^{23}(i+50)$$

$$= \sum_{i=1}^{23} i + \sum_{i=1}^{23} 50 = \frac{(23)(24)}{2} + (50)(23) = 1,426$$

Example 2.4.14.

$$\sum_{i=1}^{n}(i+1)(i+2) = \sum_{i=1}^{n}(i^2+3i+2) = \sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 2$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} + 2n.$$

In Example 2.4.14 we do not specify the value of the upper bound for the index. We use the general n. So, our answer is expressed in terms of n. In calculus you will work with many such sums.

Telescoping Sums

In Example 2.4.13 we saw a use for changing the bounds on the index of summation so that we could use the "special sums" formulas. Below is another example of a problem where it is useful to rewrite the index. Follow each step carefully (you still have that pencil, right?). Watch for changes in the indicies and be able to justify each sum rewrite.

Example 2.4.15. *Evaluate:* $\displaystyle\sum_{i=1}^{100}\left(\frac{1}{i} - \frac{1}{i+1}\right)$.

$$\sum_{i=1}^{100}\left(\frac{1}{i} - \frac{1}{i+1}\right) = \sum_{i=1}^{100}\frac{1}{i} - \sum_{i=1}^{100}\frac{1}{i+1} = \left(1 + \sum_{i=2}^{100}\frac{1}{i}\right) - \left(\sum_{i=1}^{99}\frac{1}{i+1} + \frac{1}{101}\right)$$

$$= 1 + \sum_{i=1}^{99}\frac{1}{i+1} - \sum_{i=1}^{99}\frac{1}{i+1} - \frac{1}{101} = 1 - \frac{1}{101} = \frac{100}{101}.$$

The sum in Example 2.4.15 is one of a particular type of sum, called a *telescoping* sum. A telescoping sum is characterized by some of its terms canceling with others, and doing so repetitively. This can be seen through another approach to Example 2.4.15:

Example 2.4.15 – Another View

$$\sum_{i=1}^{100}\left(\frac{1}{i}-\frac{1}{i+1}\right)=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots$$
$$\cdots+\left(\frac{1}{99}-\frac{1}{100}\right)+\left(\frac{1}{100}-\frac{1}{101}\right).$$

Notice that the only terms that do not cancel are 1 and $-\frac{1}{101}$. Thus, the total is $\frac{100}{101}$.

2.5 Exercises

Problems for Section 2.1

Problem 1. Divide. Write your answer two ways, as done in the text examples:

$$\textit{dividend} = (\textit{divisor})(\textit{quotient}) + \textit{remainder} \quad \text{and} \quad \frac{\textit{dividend}}{\textit{divisor}} = (\textit{quotient}) + \frac{\textit{remainder}}{\textit{divisor}}$$

(a) $(2x^3+5x^2+7x+3)\div(x+1)$ (b) $\dfrac{x^4+2x^3-x-2}{x+2}$

(c) $(5+4x^3-3x)\div(2x-3)$ (d) $\dfrac{x^6+3x^5-x^4+x^2+2}{x^2-2}$

(e) $\dfrac{x^4+2x^3+6x^2-1}{3x^2+6x}$ (f) $(8x^4+7)\div(2x^3+x^2-x+5)$

Problems for Section 2.2

Problem 1. For each of the following, what constant term is needed to form a perfect square?

(a) x^2+8x+ ___ (b) x^2-20x+ ___ (c) x^2+7x+ ___

Problem 2. For each of the following, what x term is needed to form a perfect square?

(a) x^2+ ___ $+16$ (b) x^2- ___ $+16$ (c) x^2+ ___ $+\frac{1}{4}$

Problem 3. Write each of the following as the sum of a perfect square and a constant.

(a) x^2+4x-7 (b) x^2-3x+2 (c) $-x^2+30x-100$
(d) $-x^2-x+1$ (e) $4x^2+24x+3$ (f) $-5x^2-10x+1$
(g) $-3x^2+12x-5$ (h) $\frac{1}{2}x^2+x+2$ (i) $9x^2+12x+4$

Problem 4. In Problem 3 above you took expressions in the form ax^2+bx+c and rewrote them into the form $a(x-h)^2+k$. For problems (a), (d), (g) and (i), identify the numbers that correspond to a, b, c, h, and k.

Problem 5. Rewrite the general expression ax^2+bx+c into $a(x-h)^2+k$ form. Show that $h=\frac{-b}{2a}$ and $k=\left(-\frac{b^2}{4a}+c\right)$

Problems for Section 2.3

Problem 1. Evaluate:

(a) $0!$ (b) $1!$ (c) $6!$ (d) $\dfrac{10!}{5!\,7!}$ (e) $\dfrac{3!\,5!}{6!\,4!}$ (f) $\dfrac{8!}{2^8}$ (g) $\dfrac{n!}{(n-2)!}$

Problem 2. Evaluate:

(a) $\binom{4}{2}$ (b) $\binom{8}{5}$ (c) $\binom{12}{7}$ (d) $\binom{20}{17}$ (e) $\binom{100}{99}$ (f) $\binom{53}{0}$

Problem 3. Without copying from the book, construct Pascal's Triangle through nine rows.

1. Circle the number in the triangle that is the coefficient for the a^2b^4 term in $(a+b)^6$.
2. Put a box around the number in the triangle that is represented by $\binom{7}{5}$.
3. Check your answer to part (b) by evaluating $\binom{7}{5}$ using factorials.
4. In the expansion of $(a+b)^8$, which two terms have coefficient 28?

Problem 4. Use the Binomial Theorem and Pascal's Triangle to expand and simplify:

(a) $(a+b)^4$ (b) $(a+b)^6$ (c) $(a-b)^6$ (d) $(a+2)^5$
(e) $(x-1)^{10}$ (f) $(3x+y)^3$ (g) $(x-2y)^4$

Problem 5. Find the following terms. Simplify.

(a) Eighth term in $(a+b)^{10}$ (b) Fourth term in $(x-y)^7$
(c) Second term in $(2x+5)^4$ (d) Third term in $(x^2+y^3)^5$
(e) Twelfth term in $(5x+1)^{13}$ (f) Last term in $(43x^2y^4z-1)^{100}$

Problem 6. Find the coefficient for the following terms:

(a) a^2b^7 term in $(a+b)^9$ (b) x^3 term in $(x+1)^7$
(c) x^3 term in $(2+x)^7$ (d) a^3b^3 term in $(2a+b)^6$

Problem 7. Add: $(a+1)^3+(a+1)^4$.

Problems for Section 2.4

Problem 1. Write each sum in expanded form. You do not have to simplify.

(a) $\displaystyle\sum_{i=1}^{5}\sqrt{i}$ (b) $\displaystyle\sum_{i=0}^{4}\frac{2i-1}{2i+1}$ (c) $\displaystyle\sum_{i=4}^{6}i^3$

(d) $\displaystyle\sum_{i=0}^{n-1}(-1)^i$ (e) $\displaystyle\sum_{i=n}^{n+3}i^3$ (f) $\displaystyle\sum_{i=3}^{6}\sqrt{8}$

Problem 2. Write in sigma (Σ) notation:

(a) $2+3+4+5+6$ (b) $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ldots+\frac{19}{20}$ (c) $2-4+6-8+\ldots-56$

(d) $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots+\frac{1}{100}$ (e) $1+x+x^2+x^3+\ldots+x^n$

Problem 3. Write an equivalent sum that has $m=0$ for the lower bound on the index of summation. Then do the same problem with $m=5$.

(a) $\sum_{i=1}^{7}(3i^2+2)$ (b) $\sum_{i=3}^{5}\frac{i-2}{3i+1}$ (c) $\sum_{i=6}^{10}(-4i+10)^2$

Problem 4. Find the numeric value for each sum. Skip problem (d) if you have not done the Binomial Theorem section.

(a) $\sum_{i=4}^{8}(3i-2)$ (b) $\sum_{i=0}^{4}3^i$ (c) $\sum_{i=0}^{4}(2^i+i^2)$ (d) $\sum_{i=0}^{5}\binom{5}{i}$

(e) $\sum_{i=1}^{10}7$ (f) $\sum_{i=1}^{5}i^3$ (g) $\sum_{i=1}^{15}i$ (h) $\sum_{i=20}^{75}i$

(i) $\sum_{i=1}^{15}3i^2$ (j) $\sum_{i=1}^{8}(i^2-i)$ (k) $\sum_{i=1}^{20}(i+1)(i-1)$

Problem 5. Find the value of n such that $\sum_{i=1}^{n}i=78$

Problem 6. Show that $\left(\sum_{i=1}^{4}i\right)\left(\sum_{i=1}^{4}i^2\right)\neq\sum_{i=1}^{4}i^3$

Is it true for all n that $\left(\sum_{i=1}^{n}i\right)\left(\sum_{i=1}^{n}i^2\right)\geq\sum_{i=1}^{n}i^3$? Justify your claim.

Problem 7. Rewrite as an expression in terms of n. Simplify. Be careful to notice the bounds on the index of summation for each problem.

(a) $\sum_{i=1}^{n}2i$ (b) $\sum_{i=1}^{n}6$ (c) $\sum_{i=1}^{n}(2-5i)$

(d) $\sum_{i=1}^{n}(i+1)(i+2)$ (e) $\sum_{i=0}^{n}(i+1)(i+2)$ (f) $\sum_{i=4}^{n}(i^2-i+3)$

Problem 8. Evaluate the following telescoping sums. You do not have to simplify.

(a) $\sum_{i=1}^{50}(3^i-3^{i+1})$ (b) $\sum_{i=4}^{79}\left(\frac{1}{i+1}-\frac{1}{i}\right)$

2.6 Answers to Exercises

Answers for Section 2.1 Exercises

Answer to Problem 1.

(a) $2x^3 + 5x^2 + 7x + 3 = (x+1)(2x^2 + 3x + 4) - 1$

$$\frac{2x^3 + 5x^2 + 7x + 3}{x+1} = 2x^2 + 3x + 4 + \frac{-1}{x+1}$$

(b) $x^4 + 2x^3 - x - 2 = (x+2)(x^3 - 1)$

$$\frac{x^4 + 2x^3 - x - 2}{x+2} = x^3 - 1 \text{ where } x \neq -2$$

(c) $5 + 4x^3 - 3x = (2x-3)(2x^2 + 3x + 3) + 14$

$$\frac{5 + 4x^3 - 3x}{2x-3} = 2x^2 + 3x + 3 + \frac{14}{2x-3}$$

(d) $x^6 + 3x^5 - x^4 + x^2 + 2 = (x^2 - 2)(x^4 + 3x^3 + x^2 + 6x + 3) + (12x + 8)$

$$\frac{x^6 + 3x^5 - x^4 + x^2 + 2}{x^2 - 2} = x^4 + 3x^3 + x^2 + 6x + 3 + \frac{12x + 8}{x^2 - 2}$$

(e) $x^4 + 2x^3 + 6x^2 - 1 = (3x^2 + 6x)(\frac{1}{3}x^2 + 2) + (-12x - 1)$

$$\frac{x^4 + 2x^3 + 6x^2 - 1}{3x^2 + 6x} = \frac{1}{3}x^2 + 2 + \frac{-12x - 1}{3x^2 + 6x}$$

(f) $8x^4 + 7 = (2x^3 + x^2 - x + 5)(4x - 2) + (6x^2 - 22x + 17)$

$$\frac{8x^4 + 7}{2x^3 + x^2 - x + 5} = 4x - 2 + \frac{6x^2 - 22x + 17}{2x^3 + x^2 - x + 5}$$

Answers for Section 2.2 Exercises

Answer to Problem 1.

(a) 16 (b) 100 (c) $\frac{49}{4}$

Answer to Problem 2.

(a) $8x$ (b) $8x$ (c) x

Answer to Problem 3.

(a) $(x+2)^2-11$	(b) $(x-\frac{3}{2})^2-\frac{1}{4}$	(c) $-(x-15)^2+125$
(d) $-(x+\frac{1}{2})^2+\frac{5}{4}$	(e) $4(x+3)^2-33$	(f) $-5(x+1)^2+6$
(g) $-3(x-2)^2+7$	(h) $\frac{1}{2}(x+1)^2+\frac{3}{2}$	(i) $9(x+\frac{2}{3})^2$

Answer to Problem 4.

(a)	$a=1$	$b=4$	$c=-7$	$h=-2$	$k=-11$
(d)	$a=-1$	$b=-1$	$c=1$	$h=-\frac{1}{2}$	$k=\frac{5}{4}$
(g)	$a=-3$	$b=12$	$c=-5$	$h=2$	$k=7$
(i)	$a=9$	$b=12$	$c=4$	$h=-\frac{2}{3}$	$k=0$

Answer to Problem 5.
N/A

Answers for Section 2.3 Exercises

Answer to Problem 1.

(a) 1 (b) 1 (c) 720 (d) 6 (e) $\dfrac{1}{24}$ (f) $\dfrac{315}{2}$ (g) $n(n-1)$

Answer to Problem 2.

(a) 6 (b) 56 (c) 792 (d) 1,140 (e) 100 (f) 1

Answer to Problem 3.

The first seven rows of Pascal's Triangle can be found on page 47.
Eighth row: $1, 7, 21, 35, 35, 21, 7, 1$. Ninth row: $1, 8, 28, 56, 70, 56, 28, 8, 1$.
(a) the second 15 in the seventh row (b) the second 21 in the eighth row
(c) 21 (d) a^6b^2 and a^2b^6.

Answer to Problem 4.

(a) $a^4+4a^3b+6a^2b^2+4ab^3+b^4$
(b) $a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$
(c) $a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$
(d) $a^5+10a^4+40a^3+80a^2+80a+32$
(e) $x^{10}-10x^9+45x^8-120x^7+210x^6-252x^5+210x^4-120x^3+45x^2-10x+1$
(f) $27x^3+27x^2y+9xy^2+y^3$
(g) $x^4-8x^3y+24x^2y^2-32xy^3+16y^4$

Answer to Problem 5.

(a) $120a^3b^7$ (b) $-35x^4y^3$ (c) $160x^3$ (d) $10x^6y^6$ (e) $1,950x^2$ (f) 1

Answer to Problem 6.

(a) 36 (b) 35 (c) 560 (d) 160

Answer to Problem 7.

$a^4 + 5a^3 + 9a^2 + 7a + 2$

Answers for Section 2.4 Exercises

Answer to Problem 1.

(a) $\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5}$

(b) $\frac{-1}{1} + \frac{1}{3} + \frac{3}{5} + \frac{5}{7} + \frac{7}{9}$

(c) $4^3 + 5^3 + 6^3$

(d) $1 - 1 + 1 - 1 + \ldots + (-1)^{n-1}$

(e) $n^3 + (n+1)^3 + (n+2)^3 + (n+3)^3$

(f) $\sqrt[3]{8} + \sqrt[4]{8} + \sqrt[5]{8} + \sqrt[6]{8}$

Answer to Problem 2.
Answers are not unique. Some solutions are:

(a) $\sum_{i=2}^{6} i$ (b) $\sum_{i=1}^{19} \frac{i}{i+1}$ (c) $\sum_{i=1}^{28} ((-1)^{i+1} 2i)$ (d) $\sum_{i=1}^{10} \frac{1}{i^2}$ (e) $\sum_{i=0}^{n} x^i$

Answer to Problem 3.

(a) $\sum_{i=0}^{6} (3(i+1)^2 + 2)$ $\sum_{i=5}^{11} (3(i-4)^2 + 2)$ (b) $\sum_{i=0}^{2} \frac{(i+3)-2}{3(i+3)+1}$ $\sum_{i=5}^{7} \frac{(i-2)-2}{3(i-2)+1}$

(c) $\sum_{i=0}^{4} (-4(i+6)+10)^2$ $\sum_{i=5}^{9} (-4(i+1)+10)^2$

Answer to Problem 4.

(a) 80 (b) 121 (c) 61 (d) 32 (e) 70 (f) 225
(g) 120 (h) 2,660 (i) 3,720 (j) 168 (k) 2,850

Answer to Problem 5.
12

Answer to Problem 6.
$300 \neq 100$ Yes

Answer to Problem 7.

(a) $n(n+1)$ (b) $6n$ (c) $\frac{-5n^2 - n}{2}$

(d) $\frac{n^3 + 6n^2 + 11n}{3}$ (e) $2 + \frac{n^3 + 6n^2 + 11n}{3}$ (f) $\frac{n^3 + 8n - 51}{3}$

Answer to Problem 8.

(a) $3 - 3^{51}$ (b) $-\frac{19}{80}$