Chapter 10

Exponential and Logarithmic Functions

In this chapter we look at two new kinds of functions, the exponential function and the logarithmic function. We begin with the exponential function.

10.1 Exponential Functions

10.1.1 Definition

In an exponential function, the independent variable is in the exponent.

Definition 10.1.1. *An exponential function is a function in the form $f(x) = a^x$ where a is a constant real number and $a > 0$ and $a \neq 1$. The number a is called the base of the function.*

The domain for this function is $\Re$, the set of all real numbers.

So, $f(x) = 2^x$ and $g(x) = (\frac{3}{7})^x$ and $h(x) = (\sqrt{5})^x$ are all examples of exponential functions.

Why do you suppose that we do not allow a to have the value 1? Consider what some of the values would be for $f(x) = 1^x$: $f(2) = 1^2 = 1$, and $f(\frac{1}{2}) = 1^{\frac{1}{2}} = 1$, and $f(0) = 1^0 = 1$. Indeed, for any value of x, $f(x) = 1$. This then is a linear function, the function $f(x) = 1$, so we do not include it in the class of exponential functions.

Why do you suppose that we do not allow a to be a negative number? One obvious reason is that we would have trouble with some x values less than 1. If $a = -4$ and $x = \frac{1}{2}$ we would not be able to evaluate $a^x = (-4)^{\frac{1}{2}} = \sqrt{-4}$. Certainly $\sqrt{-4}$ is undefined in $\Re$. So, we keep our life simple and only allow a to be a positive number, but not equal to 1.

10.1.2 Characteristics and Graphs

A Close Look at $f(x) = 2^x$

Let's look in detail at the particular exponential function $f(x) = 2^x$. We'll start by looking at some specific values of the function:

- Some positive integer values for x: $f(1) = 2^1 = 2$; $f(2) = 2^2 = 4$; $f(3) = 2^3 = 8$; $f(4) = 16$; $f(10) = 1,024$; $f(20) = 1,048,576$; $f(50)$ is a really big number with 16 digits. For $x > 10$ the use of a calculator is forgivable.

- We certainly don't want to forget our y-intercept, where $x = 0$: $f(0) = 2^0 = 1$.

- And now some negative integer values for x: $f(-1) = 2^{-1} = \frac{1}{2}$; $f(-2) = 2^{-2} = \frac{1}{4}$. Indeed we recall that $a^{-p} = \frac{1}{a^p}$ so we can use our work from above to easily get: $f(-3) = \frac{1}{f(3)} = \frac{1}{8}$; $f(-4) = \frac{1}{16}$; $f(-10) = \frac{1}{1,024}$; $f(-20) = \frac{1}{1,048,576}$; and $f(-50)$ is a fraction with a 1 in the numerator and a really big number with 16 digits in the denominator.

- Now let's look at some non-integer values of x: $f(\frac{1}{2}) = 2^{\frac{1}{2}} = \sqrt{2} \approx 1.414$. Similarly, $f(\frac{1}{3}) = 2^{\frac{1}{3}} = \sqrt[3]{2} \approx 1.260$; $f(\frac{1}{4}) = \sqrt[4]{2} \approx 1.189$; $f(\frac{1}{10}) = \sqrt[10]{2} \approx 1.072$. We use a calculator here to get decimal approximations because we will use them in our study below.

- We use the same "reciprocal trick" (and a calculator) to get function values for x's that are negative non-integer values: $f(-\frac{1}{2}) = \frac{1}{\sqrt{2}} \approx 0.707$; $f(-\frac{1}{3}) = \frac{1}{\sqrt[3]{2}} \approx 0.794$; $f(-\frac{1}{4}) \approx 0.841$; $f(-\frac{1}{10}) \approx 0.933$.

We now summarize in a list all of the data we have determined so far.

x	$f(x) = 2^x$	x	$f(x) = 2^x$
-20	$\frac{1}{1,048,576}$	$\frac{1}{10}$	1.072
-10	$\frac{1}{1,024}$	$\frac{1}{4}$	1.189
-4	$\frac{1}{16}$	$\frac{1}{3}$	1.260
-3	$\frac{1}{8}$	$\frac{1}{2}$	1.414
-2	$\frac{1}{4}$	1	2
-1	$\frac{1}{2}$	2	4
$-\frac{1}{2}$	0.707	3	8
$-\frac{1}{3}$	0.794	4	16
$-\frac{1}{4}$	0.841	10	1,024
$-\frac{1}{10}$	0.933	20	1,048,576
0	1		

Look carefully at this list of function values. We will make some observations. You should decide if you think that the observation is just peculiar to the specific data chosen, or if it will hold for the whole function $f(x) = 2^x$ or if this observation might hold for all exponential functions.

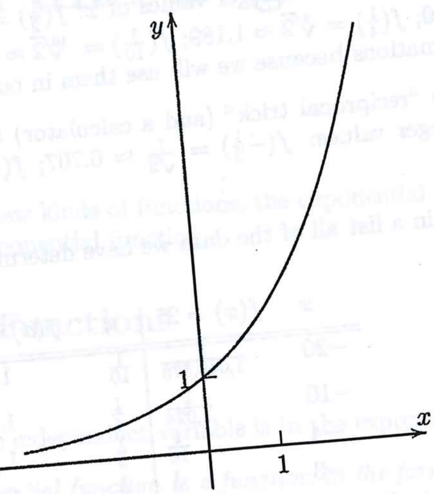
- First we observe that all of the $f(x)$ values are positive. Is there a value of x that would make 2^x negative? or make 2^x zero? Can you think of a value for a and a value for x so that a^x is negative? or zero? Remember that $a > 0$ and $a \neq 1$.

- Next we observe that the $f(x)$ values are increasing as x increases. When x is the smallest number in the list, $f(x)$ is the smallest number. When x is the largest number, $f(x)$ is the largest number. Do you think this pattern holds for x values not on the list? Is it reasonable that the value of 2^x should increase if x increases? Do you think this holds for a^x?

- We see that $f(0) = 1$. Is this true for all values of a? Is it true that $a^0 = 1$ for all values of a where $a > 0$ and $a \neq 1$?

- Finally, look at the pattern of values as x gets very large or x gets very small. What do you suppose happens to $f(x)$? We have already seen that $f(x) = 2^x$ is increasing, but is it doing so without bound or can we expect its graph to have a horizontal asymptote as $x \to \infty$? What about as $x \to -\infty$? Can you make a general statement about a^x?

We can use (some of) the data above to draw a sketch of the graph of $f(x) = 2^x$ (below). It has been stated earlier that graphing a function is more than just plotting a few points and then "playing dot-to-dot." However, it does turn out this time that the graph for this function is indeed a smooth curve that behaves nicely (it is all connected; doesn't jump around or have "holes" in it).



$y = 2^x$

Irrational Exponents

We have said that the domain for $f(x) = 2^x$ is $\Re$. Yet all of the data points that we plotted were rational x values. We had a variety of values: positive, negative, integer, non-integer. From our study of exponents in Chapter 1 we know how to deal with these. But, how do we deal with x values that are irrational, such as $x = \sqrt{3}$ or $x = \pi$? What does 2^π mean? It does not mean 2 multiplied times itself π times. It does not mean the "πth" root of 2. We can use the graph of $f(x) = 2^x$ to get a value for 2^π. 2^π is simply the y-value on the graph when the x value is π.

You might think that this is a "cheating" way to answer the question, "What do we mean by 2^π?" But, at least it is a reasonable answer. Since the graph of $f(x) = 2^x$ is smooth and increasing, and since $3.1 < \pi < 3.2$, we would want the value of 2^π to be somewhere between the values of $2^{3.1}$ and $2^{3.2}$. We know how to interpret $2^{3.1}$ and $2^{3.2}$ because these exponents are rational. Since $3.1 = \frac{31}{10}$ and $3.2 = \frac{32}{10}$ we understand $2^{3.1} = 2^{\frac{31}{10}} = \sqrt[10]{2^{31}} \approx 8.574$ and $2^{3.2} = 2^{\frac{32}{10}} = \sqrt[10]{2^{32}} \approx 9.198$. Our smooth, connected graph tells us that there IS a value for 2^π and we have figured that it must be between 8.574 and 9.198. We can get closer to the actual value of 2^π by simply choosing rational numbers closer to π than are 3.1 and 3.2. With calculus we have a way of squeezing the interval so closely around π that we say we can know the actual value of 2^π.

So, what is the point of all of this? Certainly from a practical standpoint we will simply use a calculator to get $2^\pi \approx 8.825$. But what we have here is a way to interpret an irrational exponent. The smooth connectedness of the graph of $f(x) = a^x$ gives us a way to understand the values of numbers in the form a^x where the exponent is irrational. We can consider all irrational exponents in the same way that we did here with π.

Exponential Functions with base $a > 1$

Now we will look at some exponential functions besides just $f(x) = 2^x$. Complete the following table and use it to sketch the graphs of $f(x) = 2^x$, $g(x) = 3^x$ and $h(x) = 4^x$ on the same set of axes.

x	$f(x) = 2^x$	$g(x) = 3^x$	$h(x) = 4^x$
-3	$\frac{1}{8}$		
-2	$\frac{1}{4}$		
-1	$\frac{1}{2}$		
$-\frac{1}{2}$	0.707		
0	1		
$\frac{1}{2}$	1.414		
1	2		
2	4		
3	8		

Compare the three graphs:

- When $x < 0$, which function has the highest function values? which has the lowest?
- When $x = 0$, what do you notice?
- When $x > 0$, which function has the highest function values? which has the lowest?

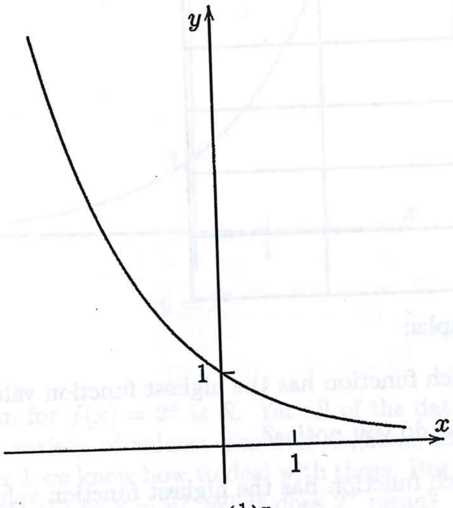
Make a general statement to describe your findings.

Use this statement to sketch in the graph of $k(x) = (\frac{7}{2})^x$ without plotting points.

Exponential Functions with base $0 < a < 1$

You will recall that the definition of exponential function allows for the base a to be any positive number except 1. So far we have dealt only with a values greater than 1. Let's consider the exponential function $f(x) = (\frac{1}{2})^x$. If we make a table of values for this function and look at its graph we see that it is a little different from the ones studied previously, although some of the numbers look quite familiar.

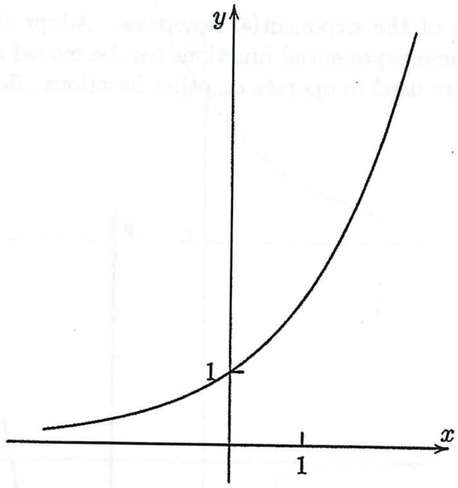
x	$f(x) = (\frac{1}{2})^x$	x	$f(x) = (\frac{1}{2})^x$
-20	$1,048,576$	$\frac{1}{10}$	0.933
-10	$1,024$	$\frac{1}{4}$	0.841
-4	16	$\frac{1}{3}$	0.794
-3	8	$\frac{1}{2}$	0.707
-2	4	1	$\frac{1}{2}$
-1	2	2	$\frac{1}{4}$
$-\frac{1}{2}$	1.414	3	$\frac{1}{8}$
$-\frac{1}{3}$	1.260	4	$\frac{1}{16}$
$-\frac{1}{4}$	1.189	10	$\frac{1}{1,024}$
$-\frac{1}{10}$	1.072	20	$\frac{1}{1,048,576}$
0	1		



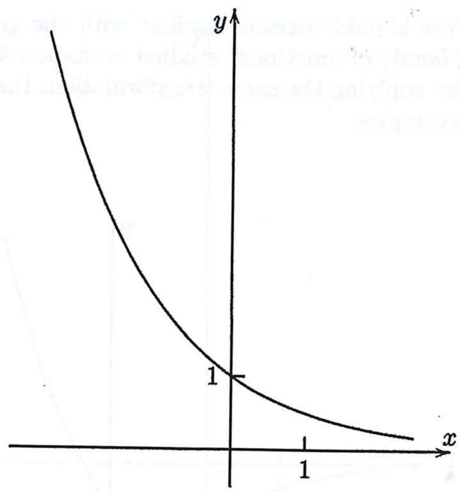
$y = (\frac{1}{2})^x$

The values for the exponential function with base $a = \frac{1}{2}$ and the values for the exponential function with base $a = 2$ are simply reciprocals of each other. We can explain this algebraically: $(\frac{1}{2})^x = \frac{1^x}{2^x} = \frac{1}{2^x}$.

We can take this a step further: $(\frac{1}{2})^x = \frac{1}{2^x} = 2^{-x}$. In Chapter 3 we learned that for any function f, the graph of $f(x)$ and the graph of $f(-x)$ are reflections of each other about the y-axis. Look at the graphs below for $f(x) = 2^x$ and $f(-x) = 2^{-x} = (\frac{1}{2})^x$



$y = 2^x$



$y = (\frac{1}{2})^x$

What do you suppose the graph of $f(x) = (\frac{1}{3})^x$ looks like? How will it compare to exponential graphs with bases of 3 or $\frac{1}{2}$?. Test your thoughts by sketching (you can do it now without plotting points) the graphs for 2^x and 3^x and then sketching in the graphs for $(\frac{1}{2})^x$ and $(\frac{1}{3})^x$ by using the reflections. Be careful. The only place that any graph will cross another is at the point $(0, 1)$.

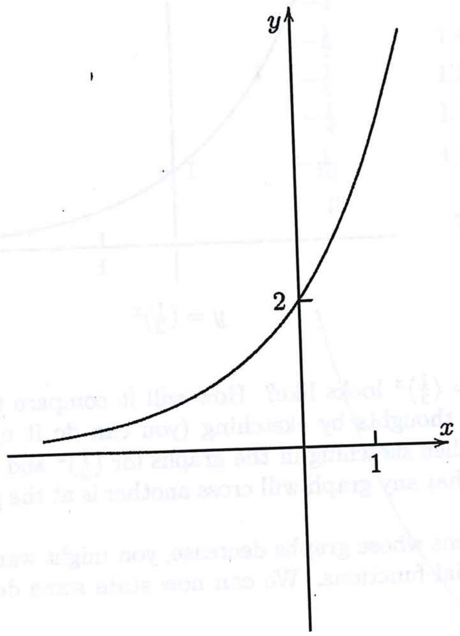
Since we now have some exponential functions whose graphs decrease, you might want to revisit the conclusions you drew concerning exponential functions. We can now state some definite facts about exponential functions.

Important Idea 10.1.1.
Functions in the form $f(x) = a^x$ where $a > 0$ and $a \neq 1$:

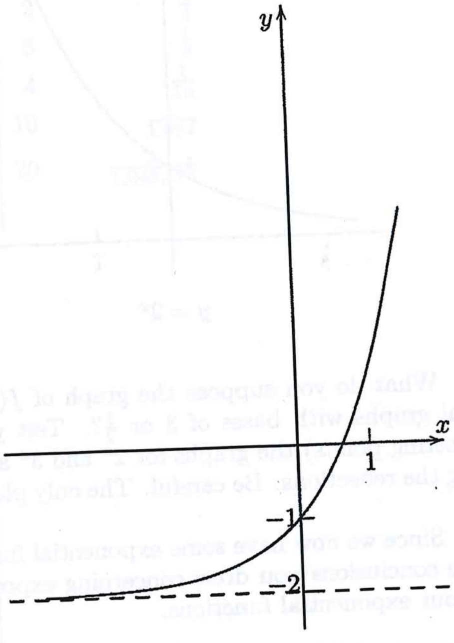
1. *have domain $\Re$,*
2. *have range $(0, \infty)$ (the function values are always positive),*
3. *have y-intercept $(0, 1)$,*
4. *and, when $a > 1$, we have:*
   (a) *f is increasing,*
   (b) *as $x \to \infty$, $f \to \infty$, and*
   (c) *as $x \to -\infty$, $f \to 0$ (there is a left horizontal asymptote $y = 0$),*
5. *and, when $0 < a < 1$ we have:*
   (a) *f is decreasing,*
   (b) *as $x \to -\infty$, $f \to \infty$, and*
   (c) *as $x \to \infty$, $f \to 0$ (there is a right horizontal asymptote $y = 0$).*

Variations

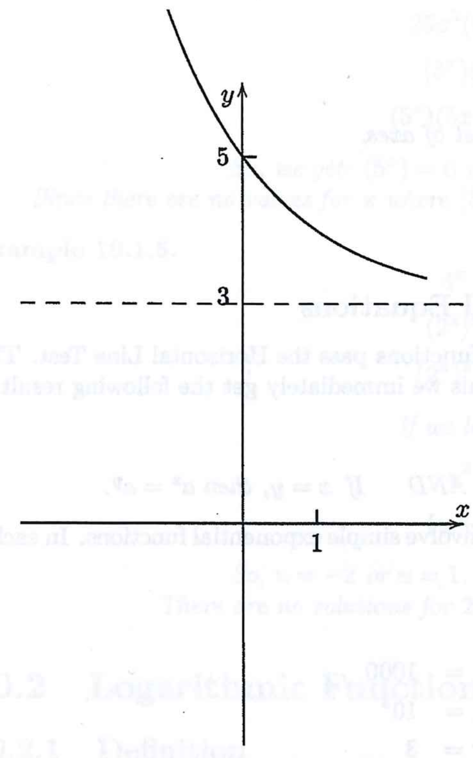
You should become familiar with the graphs of the exponential functions. Adopt them into the "family of functions" studied in chapter 4. These exponential functions can be moved and stretched by applying the same transformations that were used to operate on other functions. Below are some examples.



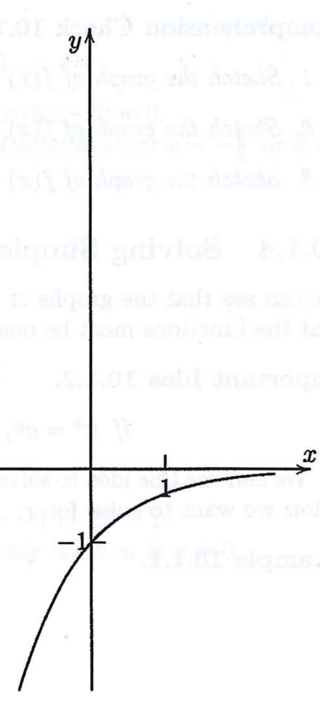
$y = 2^{x+1}$



$y = 3^x - 2$



$y = (\frac{1}{2})^{(x-1)} + 3$



$y = -(\frac{1}{3})^x$

10.1.3 The number 'e'

You are familiar with the number π. While you might not know how the actual value was arrived at you do know that it is the number that gives the ratio of the length of the arc of a semi-circle to the radius of the circle. You know that while π is irrational it can be approximated to any desired level of accuracy. In this course we are usually content with $\pi \approx 3.14$.

We now introduce another helpful irrational number. This number is called 'e.' The geometric development of e is beyond the scope of this course[1] but you will enjoy seeing its development in calculus. This number is mentioned in this course so that when you encounter it in calculus you will be comfortable with it, as you now are with π. The number e has its applications mainly with exponential and logarithmic functions.

The number $e \approx 2.71828$. Just as you think of π as being a little more than 3, you can think of e as being a little less than 3. So, now when you are asked to "pick a number between 1 and 10" you have another alternative: e. While e is a truly wonderful number, it is still a number, bound by all the same algebraic rules as any other constant, so don't let its appearance bother you.

Since e is a real number and $e > 0$ and $e \neq 1$ we can use it as the base for our exponential function.

You know what the graphs of $f(x) = 2^x$ and $f(x) = 3^x$ look like. On the blank page opposite, draw these graphs on the same set of axes. You know that $2 < e < 3$. Use this information to sketch

[1]Author's irrelevant note: I think I've read the phrase "beyond the scope of this course" in every math book I've ever read. I've always wanted to be able to write that myself.

the graph of $f(x) = e^x$.

Comprehension Check 10.1.

1. *Sketch the graph of* $f(x) = e^x$ *on a new set of axes.*
2. *Sketch the graph of* $f(x) = e^{-x}$.
3. *Sketch the graph of* $f(x) = e^{-x} + 1$.

10.1.4 Solving Simple Exponential Equations

We can see that the graphs of the exponential functions pass the Horizontal Line Test. This means that the functions must be one-to-one. From this we immediately get the following result:

Important Idea 10.1.2.

If $a^x = a^y$, *then* $x = y$. *AND* *If* $x = y$, *then* $a^x = a^y$.

We can use this idea to solve equations that involve simple exponential functions. In each example below we want to solve for x.

Example 10.1.1.

$$\begin{aligned} 10^x &= 1000 \\ 10^x &= 10^3 \\ x &= 3 \end{aligned}$$

Example 10.1.2.

$$\begin{aligned} 2^x &= \frac{1}{4} \\ 2^x &= 2^{-2} \\ x &= -2 \end{aligned}$$

Example 10.1.3.

$$\begin{aligned} 2^{3x+1} &= \sqrt{2} \\ 2^{3x+1} &= 2^{\frac{1}{2}} \\ 3x+1 &= \frac{1}{2} \\ 3x &= -\frac{1}{2} \\ x &= -\frac{1}{6} \end{aligned}$$

Sometimes we will need to rewrite the exponential term so that it can be used with previously learned equation solving methods. The next two examples use The Property of Zero; the second one is a quadratic "in disguise."

Example 10.1.4.

$$x^2(5^{x+2}) - 9(5^x) = 0$$
$$x^2(5^2 5^x) - 9(5^x) = 0$$

$$25x^2(5^x) - 9(5^x) = 0$$
$$(5^x)(25x^2 - 9) = 0$$
$$(5^x)(5x + 3)(5x - 3) = 0$$

So, we get: $(5^x) = 0$ *or* $(5x + 3) = 0$ *or* $(5x - 3) = 0$.
Since there are no values for x *where* $(5^x) = 0$, *our final solutions are* $x = -\frac{3}{5}$ *or* $x = \frac{3}{5}$.

Example 10.1.5.

$$4^x + 2^x - 2 = 0$$
$$(2^2)^x + 2^x - 2 = 0$$
$$(2^x)^2 + (2^x) - 2 = 0$$

If we let $u = 2^x$ *we get:*

$$u^2 + u - 2 = 0$$
$$(u + 2)(u - 1) = 0$$

So, $u = -2$ *or* $u = 1$. *This gives:* $2^x = -2$ *or* $2^x = 1$.
There are no solutions for $2^x = -2$, *so our only solution is* $x = 0$.

10.2 Logarithmic Functions

10.2.1 Definition

We notice that the graph of the exponential function $f(x) = a^x$ passes the Horizontal Line Test. Therefore, it is a one-to-one function and so has an inverse function, $f^{-1}(x)$.

The definition of inverse function tells us that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. For $f(x) = a^x$ this means $a^{f^{-1}(x)} = x$ and $f^{-1}(a^x) = x$. This inverse funtion for the exponential function has a special name. It is called the logarithm function. We write it as $f^{-1}(x) = \log_a(x)$ and read it "log, base a, of x." Using this notation, $a^{f^{-1}(x)} = x$ and $f^{-1}(a^x) = x$ become the following important statements:

Important Idea 10.2.1.
$a^{\log_a x} = x$ *AND* $\log_a(a^x) = x$

The notation for the logarithm function is rather strange and we will get back to it, but first let us look at some of the characteristics of the logarithm function. We can make some statements simply from what we already know about exponential functions and what we know about inverse functions in general. Recall that if two functions are inverses of each other then if one contains the ordered pair (x, y) then the other contains the ordered pair (y, x). This leads directly to some facts.

Important Idea 10.2.2.
The following are true concerning the logarithm function:

1. *The domain for* $\log_a x$ *is* $(0, \infty)$ *because* $(0, \infty)$ *is the range for* a^x.

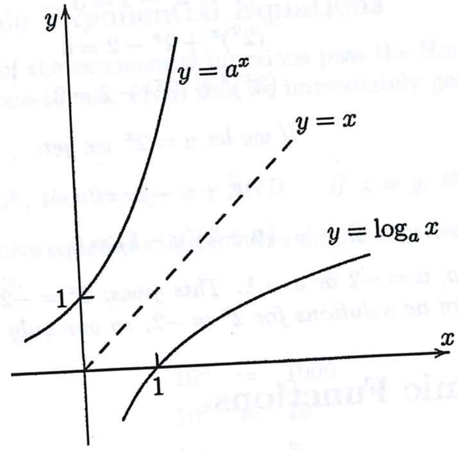
2. *The range for* $\log_a x$ *is* $\Re$ *because* $\Re$ *is the domain for* a^x.

3. $\log_a x$ *contains the point* $(1, 0)$ *because* a^x *contains the point* $(0, 1)$. *Thus, the logarithm function has an x-intercept, a root.*

4. $\log_a x$ has no y-intercept because a^x has no x-intercept.

5. $\log_a x$ has a vertical asymptote $x = 0$ because a^x has a horizontal asymptote $y = 0$.

The graph of the logarithm function is a reflection of the graph of the corresponding (same base) exponential function about the line $y = x$. Below we show the graphs for $y = a^x$ and $y = \log_a x$ when $a > 1$. It will be explained later why we do not need to be concerned separately for values of a between 0 and 1.



10.2.2 Getting a Grasp on Logarithmic Notation

Students often find the notation for logarithms to be a bit confusing. This section attempts to help you learn to work with it.

Important Idea 10.2.3.

$$\log_a x = y \textit{ means } a^y = x$$

Learn this example to help you remember: $\log_2 8 = 3$ *means* $2^3 = 8$

The two equations, $\log_a x = y$ and $a^y = x$, mean exactly the same thing. They are two ways of expressing the same relationship between x and y. The first is called the ***logarithmic*** expression and the other is called the *exponential* expression.

Important Idea 10.2.4.

A logarithm IS an exponent.

In the expresssion above, the y is *equal* to the logarithm expression. The y is also the exponent in the exponential expression. This says that the value of a logarithm function represents the exponent. So, for example, $\log_{10} 1{,}000$ must equal 3 because 3 is the exponent where if 10 were the base the resulting value would be 1,000. $10^3 = 1{,}000$

Example 10.2.1.

1. $\log_3 9 = 2$ *because* $3^2 = 9$
2. $\log_5 1 = 0$ *because* $5^0 = 1$
3. $\log_2 \frac{1}{4} = -2$ *because* $2^{-2} = \frac{1}{4}$
4. $\log_e \sqrt{e} = \frac{1}{2}$ *because* $e^{\frac{1}{2}} = \sqrt{e}$

10.2.3 Common Logs, Natural Logs and Logs with base less than 1

Logarithms can be written with any base a where $0 < a < 1$ or $a > 1$, but two bases occur so frequently that we give these logarithms special names.

A *common logarithm* is a logarithm with a base of 10. It is fairly standard notation, and we will use if from this point on, that we do not bother to explicitly write the subscript "10" when we mean a common logarithm. So, when we write "$\log x$" we will mean "$\log_{10} x$".

A *natural logarithm* is a logarithm with a base of e. This logarithm is used extensively in calculus. The shortcut special notation for the natural logarithm is to just write "ln" instead of "$\log_e$". So, we write "$\ln x$" when we mean "$\log_e x$".

For all logarithms with bases different from 10 or e it is important that you write the base specifically as the function subscript. The equations $\log 100 = 2$ and $\ln(\frac{1}{e}) = -1$ are both correct. The equation $\log 8 = 3$ is not correct. One might have *meant* a base of 2 for this last equation but what was written is a common logarithm, and it certainly isn't true that $10^3 = 8$.

Let us now address the statement made earlier that while we can have logarithms with base between 0 and 1, we only need to concern ourselves with logarithms that have base $a > 1$. Any number between 0 and 1 can be written as $\frac{1}{a}$ where a is some number greater than 1. So, suppose we have $y = \log_{\frac{1}{a}} x$ This says the same thing as $(\frac{1}{a})^y = x$. This is the same thing as $a^{-y} = x$, which in turn is the same as $\log_a x = -y$. If we combine the first and last expressions, substituting for y, we get $\log_{\frac{1}{a}} x = -\log_a x$. So, anything that we need to do with logarithms that have a base between 0 and 1 we can do by using the reciprocal base (which is greater than 1) if we negate the logarithm. Actually, if you ponder this, it shouldn't be too surprising: Logarithms are exponents and if you negate the exponent of a number what happens?

Comprehension Check 10.2.

1. *Rewrite the following logarithmic equations into their equivalent exponential forms:*
   $\log_3 81 = 4$ $\quad \log .01 = -2$ $\quad \ln e^5 = 5$ $\quad \log_7 13 = x$
2. *Rewrite the following exponential equations into their equivalent logarithmic forms:*
   $2^{-1} = \frac{1}{2}$ $\quad e^{\frac{1}{3}} = \sqrt[3]{e}$ $\quad 10^2 = 100$ $\quad 3^{20} = x$
3. *To graphically illustrate the idea that* $\log_{\frac{1}{a}} x = -\log_a x$ *we will use* $a = 2$. *Sketch the following graphs:*
   (a) *Sketch* $y = \log_{\frac{1}{2}} x$ *by first drawing* $y = (\frac{1}{2})^x$, *and then reflecting this about the line* $y = x$ *to get the desired inverse* $y = \log_{\frac{1}{2}} x$.
   (b) *Sketch* $y = -\log_2 x$ *by first drawing* $y = \log_2 x$, *and then reflecting this about the x-axis to get* $y = -\log_2 x$.
   (c) *Compare your two resulting graphs. They should be the same.*

10.2.4 Properties of Logarithms

Logarithms are exponents, so the algebra associated with logarithms follows the same rules that one uses when dealing with exponents. However, the notation for logarithms can make this somewhat difficult to see. Often it is useful to rewrite the logarithmic expression into its exponential equivalent in order to understand the properties of logarithms. Several of the most important properties are listed next, followed by an explanation and examples for each one.

Properties of Logaithms

Here we assume that a, m, n, p are values consistent with the domain of the logarithm function. So, a, m, n, p are real numbers, a, m, n are positive and $a \neq 1$.

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a a^p = p$
4. $a^{\log_a m} = m$
5. $\log_a mn = \log_a m + \log_a n$
6. $\log_a \frac{m}{n} = \log_a m - \log_a n$
7. $\log_a m^p = p \log_a m$

1. $\log_a 1 = 0$

$\log_a 1 = 0$ because $a^0 = 1$

Examples: $\log_5 1 = 0$ $\ln 1 = 0$

2. $\log_a a = 1$

$\log_a a = 1$ because $a^1 = a$

Examples: $\log_2 2 = 1$ $\ln e = 1$

3. $\log_a a^p = p$

$\log_a a^p = p$ because $a^p = a^p$

This is really just part of the original definition that the logarithm function is the inverse of the exponential function.

Examples: $\log_4 16 = \log_4 4^2 = 2$ $\ln \sqrt{e} = \frac{1}{2}$

Notice that when $p = 1$ we just have the special case that is Property 2 above.

4. $a^{\log_a m} = m$

Since $\log_a m$ represents "the exponent where if a were the base the resulting value would be m" it makes sense that if we use this $\log_a m$ as the exponent for a then our expression should be equal to m.

This is really just the other part of the original definition that the logarithm function is the inverse of the exponential function.

Examples: $6^{\log_6 17} = 17$ $e^{\ln 4} = 4$

5. $\log_a mn = \log_a m + \log_a n$

Here we will have to rewrite two logs into exponential form. So, we introduce names for them. Let $u = \log_a m$ and let $v = \log_a n$. This means that $a^u = m$ and $a^v = n$. So, the product mn is equal to $a^u a^v = a^{u+v}$. If we substitute a^{u+v} for mn in the original logarithm expression, we have $\log_a mn = \log_a a^{u+v}$ which is equal to $u + v$ (by Property 3). But $u + v$ is just $\log_a m + \log_a n$, so we have proven our claim that $\log_a mn = \log_a m + \log_a n$.

Examples: $3 = \log_2 8 = \log_2(4 \cdot 2) = \log_2 4 + \log_2 2 = 2 + 1 = 3$

$\log_4(3x) = \log_4 3 + \log_4 x$

6. $\log_a \frac{m}{n} = \log_a m - \log_a n$

This argument is left as an exercise for the student.

Examples: $3 = \log_2 8 = \log_2 \frac{32}{4} = \log_2 32 - \log_2 4 = 5 - 2 = 3$

$\log_5 \frac{2}{x} = \log_5 2 - \log_5 x$

7. $\log_a m^p = p\ \log_a m$

We will give an argument here for p a positive integer. We can give a rigorous proof for this property using an argument similar to that used for the two preceding properties. However, that argument is a tad more difficult to see and the integer argument has some instructional value so that is done here:

$$\log_a m^p = \log_a(\underbrace{m \cdot m \cdot \ldots \cdot m}_{p\ times}) = \underbrace{\log_a m + \log_a m + \ldots + \log_a m}_{p\ times} = p\ \log_a m$$

Examples: $6 = \log_2 64 = \log_2 8^2 = 2\log_2 8 = 2 \cdot 3 = 6$

$6 = \log_2 64 = \log_2 4^3 = 3\log_2 4 = 3 \cdot 2 = 6$

$\log_7 \sqrt[3]{x^2} = \log_7(x)^{\frac{2}{3}} = \frac{2}{3}\log_7 x$

As is always a good idea, we have to be careful to only use the Properties when we have the proper domains for our functions. Sometimes this requires us to be alert. For example, we could easily misuse Property 7. If simply given the function $f(x) = \log_a x^2$ we would be tempted to say that it is equal to function $g(x) = 2\log_a x$. It is not. The domain of f is $\{x : x \neq 0\}$. The domain for g is $\{x : x > 0\}$. So, these two functions are not the same. Property 7 claims equality of the expressions because it restricts the domain, only allowing m to be positive.

That having been said, in all further examples and in the homework problems we will assume that the given variables are consistent with the domains of the Properties above.

We will now do a few examples where we take a single logarithmic expression and expand it into an equivalent expression that could use multiple logarithms.

Example 10.2.2.

$$\log_2(4x^2y) = \log_2 4 + \log_2 x^2 + \log_2 y = 2 + 2\log_2 x + \log_2 y$$

Example 10.2.3.

$$\ln\left(\frac{6}{\sqrt{x^2+1}}\right) = \ln 6 - \ln\sqrt{x^2+1} = \ln 6 - \frac{1}{2}\ln(x^2+1)$$

Example 10.2.4.

$$\log\left(\frac{x^2}{y^5z^3}\right)^4 = 4\log\left(\frac{x^2}{y^5z^3}\right) = 4(\log x^2 - \log(y^5z^3)) = 4(\log x^2 - \log y^5 - \log z^3)$$

$$= 4(2\log x - 5\log y - 3\log z) = 8\log x - 20\log y - 12\log z$$

Next are a few examples where we go the other way. We take a combination of logrithmic expressions and condense them into a single expression with a coefficient of 1.

Example 10.2.5.

$$\log_3(x+2y) - \log_3(x-y) = \log_3 \frac{x+2y}{x-y}$$

Example 10.2.6.

$$\log x^2 + \frac{1}{2}\log y - \log z = \log x^2 + \log\sqrt{y} - \log z = \log \frac{x^2\sqrt{y}}{z}$$

Example 10.2.7.

$$\frac{1}{3}(\ln x - 2\ln y) + 5\ln z = \frac{1}{3}\ln\left(\frac{x}{y^2}\right) + \ln z^5$$

$$= \ln\sqrt[3]{\frac{x}{y^2}} + \ln z^5 = \ln\left(z^5\sqrt[3]{\frac{x}{y^2}}\right)$$

When using the Properties to combine or expand logarithms be careful to obey the usual rules of algebra and order of operation. Look at the following set of expressions. Make sure you understand how they are different.

Example 10.2.8.

1.

$$\frac{1}{2}\log x + \log y - \log(2z) = \log x^{\frac{1}{2}} + \log y - \log(2z)$$

$$= \log\sqrt{x} + \log y - \log(2z) = \log\frac{\sqrt{x}y}{2z}$$

2.

$$\frac{1}{2}(\log x + \log y) - \log(2z) = \frac{1}{2}\log(xy) - \log(2z)$$

$$= \log(xy)^{\frac{1}{2}} - \log(2z) = \log\sqrt{xy} - \log(2z) = \log\frac{\sqrt{xy}}{2z}$$

Or, you could use the distributive property on the $\frac{1}{2}$*:*

$$\frac{1}{2}(\log x + \log y) - \log(2z) = \frac{1}{2}\log x + \frac{1}{2}\log y - \log(2z)$$

$$= \log x^{\frac{1}{2}} + \log y^{\frac{1}{2}} - \log(2z) = \log\sqrt{x} + \log\sqrt{y} - \log(2z)$$

$$= \log\frac{\sqrt{x}\sqrt{y}}{2z} = \log\frac{\sqrt{xy}}{2z}$$

3.

$$\frac{1}{2}(\log x + \log y - \log(2z)) = \frac{1}{2}\left(\log\frac{xy}{2z}\right)$$

$$= \log\left(\frac{xy}{2z}\right)^{\frac{1}{2}} = \log\sqrt{\frac{xy}{2z}}$$

We can use the Properties of Logarithms to rewrite and simplify logarithmic expressions.

Example 10.2.9.

$$\log_6 9 + \log_6 4 = \log_6(9 \cdot 4) = \log_6 36 = 2$$

Example 10.2.10.

$$\log_9 25 - \log_9 75 = \log_9 \frac{25}{75} = \log_9 \frac{1}{3} = -\frac{1}{2}$$

Example 10.2.11.

$$\frac{2}{3}\ln 27 + 2\ln 2 - \ln 3 = \ln 27^{\frac{2}{3}} + \ln 2^2 - \ln 3$$

$$= \ln 9 + \ln 4 - \ln 3 = \ln \frac{9 \cdot 4}{3} = \ln 12$$

Comprehension Check 10.3.

1. *Explain in words the difference between* $(\log_a x)(\log_a y)$ *and* $\log_a(xy)$. *Which one is used in Property 5?*

2. *Explain in words the difference between* $\dfrac{\log_a x}{\log_a y}$ *and* $\log_a\left(\dfrac{x}{y}\right)$. *Which one is used in Property 6?*

3. *Explain in words the difference between* $(\log_a x^p)$ *and* $(\log_a x)^p$. *Which one is used in Property 7?*

10.2.5 Changing Bases

Suppose you would like to know the approximate value of x for $2^x = 100$. You know that the number is somewhere between 6 and 7 (why)?, but you need to be more precise than that. You are looking for an exponent value. So, you are looking for a logarithm. In particular, you want to know the value of $\log_2 100$. This is good so far. But when you then go to your calculator you realize that it doesn't handle logarithms with a base of 2. The only logarithm buttons on your calculator are "log" (common logs, base 10) and "ln" (natural logs, base e). You need to be able to change a base 2 logarithm into a base 10 or base e logarithm.

There is a straightforward way to change from one base to another. It uses the algebra that we already know for logarithms, and the following fact:

Important Idea 10.2.5.

If $x = y$, *then* $\log_a x = \log_a y$ *AND* *If* $\log_a x = \log_a y$, *then* $x = y$.

Follow the steps as we change from a base a logarithm $\log_a x$ into an expression involving base b logarithms.

We will call our base a logarithm y. So,

$$y = \log_a x$$

$$a^y = x$$

$$\log_b a^y = \log_b x$$

$$y \log_b a = \log_b x$$

$$y = \frac{\log_b x}{\log_b a}$$

Now, substituting back for y we get:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

This is the "Change of Base Formula for Logarithms", restated below.

Change of Base Formula for Logarithms

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Look carefully at the placement of the a's, b's, x's.

For our particular example we can change $\log_2 100$ into $\dfrac{\log 100}{\log 2}$. A calculator will give approximate values: $\dfrac{2}{0.30103} \approx 6.64386$

We could also change to natural logs: $\log_2 100 = \dfrac{\ln 100}{\ln 2} \approx \dfrac{4.60517}{0.69315} \approx 6.64386.$

The intermediate values in these calculations were different but the end result was the same, and was indeed a number between 6 and 7.

In the example above we found it useful to change a base 2 logarithm into a common log or natural log. However, the change of base formula works for changing to any base. $\log_2 100$ is in fact equal to $\dfrac{\log_7 100}{\log_7 2}$, $\dfrac{\log_{13} 100}{\log_{13} 2}$, and $\dfrac{\log_{88} 100}{\log_{88} 2}$. All of these fractions are approximately 6.64386.

Example 10.2.12.
Change $\log_3 25$ into a base 7 logarithm.

Using the Change of Base for Logarithms formula we apply: $a = 3, b = 7$ *and* $x = 25$

$$\log_3 25 = \frac{\log_7 25}{\log_7 3}$$

Example 10.2.13.
Show that $\ln 10 = \dfrac{1}{\log e}$.

$$\ln 10 = \frac{\log 10}{\log e} = \frac{1}{\log e}$$

It would be reasonable to ask if there is a change of base formula for exponential expressions. In other words, if we are given the expression a^x is there some y so that $a^x = b^y$ for a desired base b? Yes. We find it by simply solving the equation $a^x = b^y$ for y.

$$a^x = b^y$$
$$\log_b a^x = \log_b b^y$$
$$x \log_b a = y$$

Thus we have the following result:

Change of Base Formula for Exponential Expressions

$$a^x = b^{x \log_b a}$$

Look carefully at the placement of the a's, b's, x's.

Since most calculators can handle expressions of various bases, the change of base formula for exponential expressions is not as critical as the change of base formula for logarithms. However, there is a great need in calculus to be able to change the base of an exponential expression to an equivalent expression that uses the base e. The general formula above becomes:

$$a^x = e^{x \ln a}$$

Example 10.2.14.
Change the function $f(x) = 2^x$ to an equivalent function that uses e as its base.

$$f(x) = 2^x = e^{\ln 2^x} = e^{x \ln 2}$$

10.2.6 Solving Logarithmic and Exponential Equations

We can use the idea that logarithm functions and exponential functions are inverses when we try to solve equations that include these functions. Just like when we use subtraction to undo addition, and we use multiplication to undo division and we use cosine to undo arccosine we will now use logarithms to undo exponential functions and vice-versa. We will start with some examples of using logarithms to solve exponential functions. In all examples we will solve for x.

Example 10.2.15.
Solve for x: $y = 3e^x$

$$y = 3e^x. \text{ So, } \left(\tfrac{y}{3}\right) = e^x.$$

Now we "take the natural log" of both sides: $\ln\left(\frac{y}{3}\right) = \ln e^x$.

But $\ln e^x = x$ *so we are finished.* $\ln\left(\frac{y}{3}\right) = x$

Notice that we used a logarithm with base e in order to solve for an x that was in an exponential expression with base e. We used the logarithm to bring the x out of the exponent. Sometimes we use this operation without really realizing it. We can mentally do the rewriting of the logarithm into its equivalent exponential form. We are still using the logarithm concept, just not writing it down or even acknowledging it. This point is illustrated in Example 10.2.16. Compare both problems in this example. Where in the first problem are we implicitly using the logarithm?

Example 10.2.16.
Solve for x: $2(1 + 4^x) = 6$ *and* $2(1 + 4^x) = 12$

$$\begin{aligned} 2(1+4^x) &= 6 \\ 1+4^x &= 3 \\ 4^x &= 2 \\ x &= \tfrac{1}{2} \end{aligned} \qquad \begin{aligned} 2(1+4^x) &= 12 \\ 1+4^x &= 6 \\ 4^x &= 5 \\ \log_4 4^x &= \log_4 5 \\ x &= \log_4 5 \end{aligned}$$

Notice that in the second problem of Example 10.2.16 we leave our answer in terms of a base 4 logarithm. While that is correct, it is not very useful for giving us an idea of the value of the answer. In Example 10.2.17 we show three ways of solving an exponential equation. All of the answers are equal. The prefered method will depend on the desired application and the capabilities of your calculator.

Example 10.2.17.
Solve for x: $8 = 4(3^{2x+1})$

$$\begin{aligned} 8 &= 4(3^{2x+1}) \\ 2 &= 3^{2x+1} \\ \log_3 2 &= \log_3(3^{2x+1}) \\ \log_3 2 &= 2x+1 \\ \log_3 2 - 1 &= 2x \\ \frac{\log_3 2 - 1}{2} &= x \end{aligned} \qquad \begin{aligned} 8 &= 4(3^{2x+1}) \\ 2 &= 3^{2x+1} \\ \log 2 &= \log(3^{2x+1}) \\ \log 2 &= (2x+1)\log 3 \\ \frac{\log 2}{\log 3} &= 2x+1 \\ \frac{\log 2}{\log 3} - 1 &= 2x \\ \frac{\frac{\log 2}{\log 3} - 1}{2} &= x \end{aligned} \qquad \begin{aligned} 8 &= 4(3^{2x+1}) \\ 2 &= 3^{2x+1} \\ \ln 2 &= \ln(3^{2x+1}) \\ \ln 2 &= (2x+1)\ln 3 \\ \frac{\ln 2}{\ln 3} &= 2x+1 \\ \frac{\ln 2}{\ln 3} - 1 &= 2x \\ \frac{\frac{\ln 2}{\ln 3} - 1}{2} &= x \end{aligned}$$

When you "take the log" of both sides of an equation you must use the same-based log on both sides. If you have an exponential equation with multiple bases then you need to decide which logarithm base you wish to use. You could choose any of the bases that are in the problem or you could choose something completely different, such as a common log or a natural log. Again, we offer an example with multiple, but equivalent, solutions.

Example 10.2.18.
Solve for x: $5^x = 6^{x-1}$

$$\begin{aligned} 5^x &= 6^{x-1} \\ \log_5 5^x &= \log_5 6^{x-1} \\ x &= (x-1)\log_5 6 \\ x &= x\log_5 6 - \log_5 6 \\ x - x\log_5 6 &= -\log_5 6 \\ x(1-\log_5 6) &= -\log_5 6 \\ x &= \frac{-\log_5 6}{1-\log_5 6} \end{aligned} \qquad \begin{aligned} 5^x &= 6^{x-1} \\ \log_6 5^x &= \log_6 6^{x-1} \\ x\log_6 5 &= x-1 \\ x\log_6 5 - x &= -1 \\ x(\log_6 5 - 1) &= -1 \\ x &= \frac{-1}{\log_6 5 - 1} \end{aligned} \qquad \begin{aligned} 5^x &= 6^{x-1} \\ \ln 5^x &= \ln 6^{x-1} \\ x\ln 5 &= (x-1)\ln 6 \\ x\ln 5 &= x\ln 6 - \ln 6 \\ x\ln 5 - x\ln 6 &= -\ln 6 \\ x(\ln 5 - \ln 6) &= -\ln 6 \\ x &= \frac{-\ln 6}{\ln 5 - \ln 6} \end{aligned}$$

Finally, we have an example that reminds us that we can still use all of the creativity of algebra. We are simply now including the logarithm to help us with the exponential pieces of equations.

Example 10.2.19.
Solve for x: $2(3^{2x}) - 3^x - 3 = 0$

$$2(3^{2x}) - 3^x - 3 = 0$$

$$2(3^x)^2 - (3^x) - 3 = 0$$

Now let "u" = (3^x)

$$2u^2 - u - 3 = 0$$

$$(2u-3)(u+1) = 0$$

$$u = \tfrac{3}{2} \quad or \quad u = -1$$

$$3^x = \tfrac{3}{2} \quad or \quad 3^x = -1$$

$x = \log_3 \frac{3}{2}$ *is the only solution because* $3^x = -1$ *has no solution.*

We now start with some logarithmic equations and will use the inverse operation of raising the expression to a power in order to solve for x.

Example 10.2.20.

Solve for x: $\log_5(x+3) = 2$

$$\log_5(x+3) = 2$$

$$5^{\log_5(x+3)} = 5^2$$

$$x + 3 = 25$$

$$x = 22$$

Notice that the base we choose for the "raising both sides" is the same as the logarithm base.

Example 10.2.21.

Solve for x: $2 + 3\ln(x-10) = 0$

$$2 + 3\ln(x-10) = 0$$

$$3\ln(x-10) = -2$$

$$\ln(x-10) = \frac{-2}{3}$$

$$e^{\ln(x-10)} = e^{\frac{-2}{3}}$$

$$x - 10 = e^{\frac{-2}{3}}$$

$$x = e^{\frac{-2}{3}} + 10$$

Notice that we isolate the logarithm before applying the "raising both sides" operation. If you have more than one logarithmic expression you should combine them first.

Example 10.2.22.

Solve for x: $\log_3 x + \log_3(x+2) = 1$

$$\log_3 x + \log_3(x+2) = 1$$

$$\log_3(x \cdot (x+2)) = 1$$

$$\log_3(x^2 + 2x) = 1$$

$$3^{\log_3(x^2+2x)} = 3^1$$

$$x^2 + 2x = 3$$

$$x^2 + 2x - 3 = 0$$

$$(x+3)(x-1) = 0$$

$x = -3$ and $x = 1$ appear to be solutions. However, $x = -3$ is not a solution because it is not in the domain of the original problem.

Important Idea 10.2.6.
When solving logarithmic equations always check your solution in the original problem to make sure that you have no domain violations. Remember that the domain for a logarithm function can only be values that make its argument positive.

In Example 10.2.22 neither $\log_3 x$ nor $\log_3(x+2)$ can accept $x = -3$ as input. However, you only need to have one given logarithmic expression undefined by your "solution" to make that "solution" invalid.

Go back to Examples 10.2.20 and 10.2.21 and make sure that the solutions presented are valid.

10.3 Exercises

Problems for Section 10.1

Problem 1. Use the methods of Section 10.1 to solve each equation for x.

(a) $2(1+4^x) = 6$ (b) $2^{x+3} = 4^{x-1}$ (c) $\frac{5^{x+3}}{5^{2x}} = 25$ (d) $x^3e^x - e^x = 0$

Problem 2. Without using a table of values and plotting points, sketch the graph of each of the following exponential functions. On your graph label the y-intercepts, and give the equation of the asymptote.

(a) $f(x) = 2^x - 3$ (b) $f(x) = \left(\frac{1}{e}\right)^x$ (c) $f(x) = -3^x + 1$ (d) $f(x) = 5^{-x}$

Problem 3. For what values of x is $\left(\frac{1}{2}\right)^x < \left(\frac{1}{3}\right)^x$? Use graphs to help you decide.

Problems for Section 10.2

Problem 1. Change each logarithmic statement to its equivalent exponential form.

(a) $5 = \log_2 x$ (b) $y = \ln 27$ (c) $12 = \log_a 5$

Problem 2. Change each exponential statement to its logarithmic equivalent form.

(a) $3^x = 2$ (b) $e^5 = y$ (c) $x^4 = 9$

Problem 3. Evaluate the following numbers without using a calculator.

(a) $\ln 1$ (b) $\log(.01)$ (c) $\log_3 81$

(d) $\log_4 2$ (e) $\log_3 \frac{1}{27}$ (f) $\log_{\frac{1}{2}} 8$

(g) $\log_4 2 + \log_4 16$ (h) $\log_2 \sqrt[3]{\sqrt[4]{2}}$ (i) $2\ln e^3 + \ln e^{-5} + e^{\ln 3}$

Problem 4. Without using a calculator, find the value of x.

(a) $\log_9 x = \frac{1}{2}$ (b) $\log_x 11 = 1$ (c) $\log_x 27 = \frac{3}{2}$ (d) $\log_6 x = -2$

Problem 5. Which is larger: $\log_6 37$ or $\log_7 48$? Why? Do not use a calculator.

Problem 6. Without making a table of values and plotting points, sketch the graph of each of the following logarithmic functions. Give the equation of the asymptote.

(a) $f(x) = \log_2 x + 3$ (b) $f(x) = \log_2(x+3)$ (c) $f(x) = -\ln x$

Problem 7. Find the domain for each of the following functions.

(a) $f(x) = \log_3(x-7) - \log_3(x+2)$ (b) $f(x) = \log 3^x$

(c) $f(x) = \ln(e-x)$ (d) $f(x) = \ln(x^2 - x - 2)$

Problem 8. Given $f(x) = \log_3(x+2)$ find its domain, find its inverse $f^{-1}(x)$, and give the range of $f^{-1}(x)$.

Problem 9. Use the Properties of Logarithms to condense each of these expressions into a single logarithmic expression with a positive exponent and a coefficient of 1.

(a) $\log_3 x + \log_3 2$ (b) $\log_2 9 - \log_2 y$ (c) $2\log x - 5\log y$

(d) $\frac{1}{2}\log_4(x+5)$ (e) $-4\log_6(2x)$ (f) $3\ln x + 4\ln y - 4\ln z$

(g) $\frac{1}{3}[\log_2 x + \log_2(x+1)]$

Problem 10. Use the Properties of Logarithms to expand each of these single logarithms into expressions with multiple logarithms having single character arguments.

(a) $\log_3\left(\frac{y}{2}\right)$ (b) $\log(10x)$ (c) $\log_6\left(\frac{1}{z^3}\right)$ (d) $\log_4(4x^2y)$

(e) $\log_4(4xy)^2$ (f) $\log\left(\frac{x^2-1}{x^3}\right)$ (g) $\ln\sqrt[5]{\frac{x^2}{y^3}}$ (h) $\log_2\frac{\sqrt{x}}{z^4}$

Problem 11. Use the Properties of Logarithms to evaluate the following:

(a) $\log_6 12 + \log_6 3 - \ln 1$ (b) $\frac{2}{3}\log_4 8 + \frac{1}{2}\log_4 9 - \log_4 6$

Problem 12.

(a) Use Property 6 to show that $\log_a \frac{1}{n} = -\log_a n$

(b) Use Property 7 to show that $\log_a \frac{1}{n} = -\log_a n$

Problem 13. Prove Property 6. Use the argument done for Property 5 as a model and make the necessary adjustments. (C'mon, do it! It's not bad).

Problem 14. In Example 10.2.13 we showed that $\ln 10 = \dfrac{1}{\log e}$.

(a) How large would you expect the number $\ln 10$ to be (make a reasonable guess, remembering that you are looking for an exponent which will give you 10 if e is the base)?
(b) How large would you expect the number $\log e$ to be (make a reasonable guess, remembering that you are looking for an exponent which will give you e if 10 is the base)?
(c) Use a calculator to find the approximate values for $\ln 10$ and $\log e$. How close were your estimates?
(d) Check (still using your calculator) that these numbers are indeed reciprocals).

Problem 15. Rewrite $\log_7 9$ as an equivalent logarithm using:

(a) base 5 (b) base 10 (c) base e

Problem 16. Use the Change of Base Formula to show that $\log_a b = \dfrac{1}{\log_b a}$

Problem 17. Use the Change of Base Formula to show that $\log_2 5 = 2\log_4 5$

Problem 18. Use the Change of Base for Exponential Expressions to change each expression below to its equivalent expression in base e.

(a) 3^x (b) 6^x (c) 10^x

Problem 19. Given: $\ln 2 \approx 0.69$ and $\ln 3 \approx 1.10$ and $\ln 5 \approx 1.61$, find approximate values for the following numbers without using a calculator:

(a) $\ln 15$ (b) $\ln(1.5)$ (c) $\ln \sqrt[3]{2}$ (d) $\ln(0.9)$

Problem 20. Given: $\log 3 \approx 0.477$, find approximate values for the following numbers without using a calculator:

(a) $\log 30$ (b) $\log 3000$ (c) $\log_3 10$

Problem 21. Given the number $\log_9 21$:

(a) Rewrite it as a ratio of common logs
(b) Rewrite it as a ratio of natural logs
(c) Show that it is equal to $\log_3 \sqrt{21}$

Problem 22. Solve the following exponential equations for x

(a) $2^{x-3} = 32$
(b) $2(5^x) = 32$
(c) $6^x + 10 = 47$
(d) $7(4^{6x-2}) + 13 = 41$
(e) $3^{2x} - 5(3^x) - 6 = 0$
(f) $4^x = 3^{2x-1}$ Use a natural log
(g) $2^x = 2(5^x)$ Use a common log

Problem 23. Solve the following logarithmic equations for x

(a) $2\log_5(3x) = 4$
(b) $3 + 2\log x = 15$
(c) $\ln(\ln x) = 2$
(d) $\log_3(x+1) - \log_3 x = 2$
(e) $\log_{12}(x-6) - \log_{12} 4 = 1$
(f) $\log_6(x+2) + \log_6(x+7) = 2$
(g) $\log_3(2x-1) = 2\log_3 x$

10.4 Answers to Exercises

Answers for Section 10.1 Exercises

Answer to Problem 1.

(a) $\frac{1}{2}$ (b) 5 (c) 1 (d) 1

Answer to Problem 2.

(a) $(0,-2)$, $y=-3$ (b) $(0,1)$, $y=0$ (c) $(0,0)$, $y=1$ (d) $(0,1)$, $y=0$

Answer to Problem 3.

$x<0$

Answers for Section 10.2 Exercises

Answer to Problem 1.

(a) $2^5=x$ (b) $e^y=27$ (c) $a^{12}=5$

Answer to Problem 2.

(a) $\log_3 2=x$ (b) $\ln y=5$ (c) $\log_x 9=4$

Answer to Problem 3.

(a) 0 (b) -2 (c) 4 (d) $\dfrac{1}{2}$ (e) -3

(f) -3 (g) $\dfrac{5}{2}$ (h) $\dfrac{1}{12}$ (i) 4

Answer to Problem 4.

(a) 3 (b) 11 (c) 9 (d) $\dfrac{1}{36}$

Answer to Problem 5.

$\log_6 37$

Answer to Problem 6.

(a) $x=0$ (b) $x=-3$ (c) $x=0$

Answer to Problem 7.

(a) $\{x\in\Re : x>7\}$ (b) $\Re$ (c) $\{x\in\Re : x<e\}$ (d) $\{x\in\Re : x<-1 \text{ or } x>2\}$

Answer to Problem 8.

$D_f = (-2, \infty)$ $\quad f^{-1}(x) = 3^x - 2 \quad R_{f^{-1}} = D_f = (-2, \infty)$.

Answer to Problem 9.

(a) $\log_3(2x)$ (b) $\log_2\left(\frac{9}{y}\right)$ (c) $\log\left(\frac{x^2}{y^5}\right)$ (d) $\log_4\sqrt{x+5}$

(e) $\log_6\left(\frac{1}{16x^4}\right)$ (f) $\ln\left(\frac{x^3y^4}{z^4}\right)$ (g) $\log_2\sqrt[3]{x^2+x}$

Answer to Problem 10.

(a) $\log_3 y - \log_3 2$ (b) $1 + \log x$ (c) $-3\log_6 z$

(d) $1 + 2\log_4 x + \log_4 y$ (e) $2 + 2\log_4 x + 2\log_4 y$ (f) $\log(x^2-1) - 3\log x$

(g) $\frac{2}{5}\ln x - \frac{3}{5}\ln y$ (h) $\frac{1}{2}\log_2 x - 4\log_2 z$

Answer to Problem 11.

(a) 2 (b) $\frac{1}{2}$

Answer to Problem 12.

(a) $\log_a\left(\frac{1}{n}\right) = \log_a 1 - \log_a n = 0 - \log_a n = -\log_a n$

(b) $\log_a\left(\frac{1}{n}\right) = \log_a n^{-1} = -1\log_a n = -\log_a n$

Answer to Problem 13.

Proof not shown.

Answer to Problem 14.

Answers will vary.

Answer to Problem 15.

(a) $\frac{\log_5 9}{\log_5 7}$ (b) $\frac{\log_{10} 9}{\log_{10} 7}$ (c) $\frac{\ln 9}{\ln 7}$

Answer to Problem 16.

$$\log_a b = \frac{\log_b b}{\log_b a} = \frac{1}{\log_b a}$$

Answer to Problem 17.

$$\log_2 5 = \frac{\log_4 5}{\log_4 2} = \frac{\log_4 5}{\frac{1}{2}} = 2\log_4 5$$

Answer to Problem 18.

(a) $e^{x\ln 3}$ (b) $e^{x\ln 6}$ (c) $e^{x\ln 10}$

Answer to Problem 19.

(a) 2.71 (b) 0.41 (c) 0.23 (d) -0.1

Answer to Problem 20.

(a) 1.477 (b) 3.477 (c) $\dfrac{1}{0.477}$

Answer to Problem 21.

(a) $\dfrac{\log 21}{\log 9}$ (b) $\dfrac{\ln 21}{\ln 9}$ (c) Proof not shown. Hint: Rewrite using base 3.

Answer to Problem 22.

(a) 8 (b) $\log_5 16$ (c) $\log_6 37$ (d) $\dfrac{1}{2}$

(e) $\log_3 6$ (f) $\dfrac{-\ln 3}{\ln 4 - 2\ln 3}$ (g) $\dfrac{\log 2}{\log 2 - \log 5}$

Answer to Problem 23.

(a) $\dfrac{25}{3}$ (b) $1,000,000$ (c) e^{e^2} (d) $\dfrac{1}{8}$ (e) 54 (f) 2 (g) 1

Chapter 11

Systems of Equations and Inequalities

11.1 Systems of Equations

General Systems

Given the equation $x^2 + y = 7$ there are many solutions. That is, there are many ordered pairs (x, y) for which this equation is true. Some solutions are $(1, 6)$, $(2, 3)$, $(-5, -18)$ and $(0, 7)$.

Given the equation $y = x + 1$, we can find many solutions to this also. $(1, 2)$, $(2, 3)$ and $(-5, -4)$ are solutions.

In this chapter we are interested in *systems of equations*. A system of equations is a set of two or more equations. The solution to a system of equations is the set of all ordered pairs that are solutions to *all* of the equations in the system. Systems of equations are sometimes called *simultaneous* equations because the equations have to be true simultaneously (at the same time). A system of equations is usually written with a large left-hand brace beside the equations. We can make a system out of the two equations above.

$$\begin{cases} x^2 + y = 7 \\ y = x + 1 \end{cases}$$

The solution to this system is all of the ordered pairs (x, y) that make both of the equations true. We see from our partial lists above that the point $(2, 3)$ is part of the solution to the system. Are there other points?

The second equation in the system tells us that in this system y is the same as $x + 1$. We use this information to rewrite the first equation.

$$x^2 + y = 7$$

$$x^2 + (x + 1) = 7$$

$$x^2 + x - 6 = 0$$

$$(x + 3)(x - 2) = 0$$

$$x = -3 \text{ or } x = 2$$

Since $y = x + 1$ we substitute our x values and get the solutions $(-3, -2)$ and $(2, 3)$.

Example 11.1.1.

Find the solution to the system: $\begin{cases} x + y = 4 \\ x^2 + y^2 - 4x = 0 \end{cases}$

We rewrite the first equation to $x = 4 - y$ and substitute it into the second equation:

$$x^2 + y^2 - 4x = 0$$

$$(4 - y)^2 + y^2 - 4(4 - y) = 0$$

$$16 - 8y + y^2 + y^2 - 16 + 4y = 0$$

$$2y^2 - 4y = 0$$

$$2y(y - 2) = 0$$

$$y = 0 \text{ or } y = 2$$

Since $x = 4 - y$ we substitute our y values and get solutions $(4, 0)$ and $(2, 2)$.

In Example 11.1.1 we could have rewritten the first equation as $y = 4 - x$ and substituted this expression for y in the second equation. Indeed, this appears to be the easier method. In theory we could have rewritten the second equation, solving for either x or y, and then substituted into the first equation to find the solution of the system. Do you see why this is unnecessarily awkward?

Comprehension Check 11.1.

Solve the system in Example 11.1.1 by solving the first equation for y instead of x. Check to be sure that you got the same solutions, $(4, 0)$ and $(2, 2)$.

Finding the solution to a system of equations means finding all of the ordered pairs (x, y) that make all of the equations true. The set of ordered pairs that makes any *one* of the equations true can be illustrated on a Cartesian coordinate system as the graph of the equation. So, in theory, we could draw the graphs of all of the equations in the system on the same set of axes and see where they intersect. This would give us all of the points in common, the solution to the system. In fact the instructions to "Find the solution of the system " means the same thing as "Find the intersection of the graphs of the equations" (of the system).

Comprehension Check 11.2.

Carefully graph the equations of the system $\begin{cases} x^2 + y = 7 \\ y = x + 1 \end{cases}$ *and verify that their intersections are* $(-3, -2)$ *and* $(2, 3)$*, the solutions obtained earlier.*

While finding the intersections of the graphs does give us the solution to the system, this is not a very practical way to find the solution to a system. To appreciate this point, carefully graph the equations of the following system. They are lines so the graphing is not difficult.

$$\begin{cases} 3x + 5y = 2 \\ 2y - x = 1 \end{cases}$$

What intersection did you find?

'his system algebraically. We rewrite the second equation to $x = 2y - 1$ and
' equation:

$$3(2y-1)+5y=2$$
$$6y-3+5y=2$$
$$11y=5$$
$$y=\frac{5}{11}$$

We substitute this value for y into $x = 2y - 1$ and get $x = -\frac{1}{11}$. Our solution is $(-\frac{1}{11}, \frac{5}{11})$. Did you get that exact point from your graph? Likely not. If you graphed carefully you probably got a point in the second quadrant, but it is hard to be precise.

In the Example 11.1.1 system that you graphed it was easy to verify that the intersection points matched the solution set. In fact, had you done the graph first, found the intersection points and tested them in the equations (just to make sure) you could have solved your system. The given system of lines was more difficult because the intersection did not have nice integer values. It is hard to "eyeball" numbers like $\frac{5}{11}$ and $-\frac{1}{11}$. But even these numbers are rational. Can you imagine how difficult it would be if you got a solution point that algebraically worked out to be $(\pi, \frac{\sqrt{5}}{7})$? So, although the theory is solid—intersection points of graphs *are* the solutions—in general graphing is not a practical way to solve these problems.

Graphing is not totally useless however. Graphing the equations of a system on the same set of axes can tell us how many solutions there are and can give us a general idea of where to look for them. This technique is very useful in calculus where being close to a solution is the first goal in some algorithmic processes.

Let's consider the system: $\begin{cases} y = \sin(2x) & \textit{where } x \in [0, 2\pi] \\ y = \cos x & \textit{where } x \in [0, 2\pi] \end{cases}$

Carefully graph each of the equations over the interval $[0, 2\pi]$.
What intersections do you see? If your graph is good, you should see that the curves cross each other at $(\frac{\pi}{2}, 0)$ and $(\frac{3\pi}{2}, 0)$. You can easily algebraically verify that these are solutions to the system. What other intersections are there? You should have one on x interval $(0, \frac{\pi}{2})$ and one on x interval $(\frac{\pi}{2}, \pi)$. You can certainly make some close guesses as to what these points are. It is left as an exercise to find their exact values.

Linear Systems

Suppose we have a system of two linear equations in x and y. How many solutions can this system have? If we think about the graphs of two lines in a plane we can get our answer. The lines might intersect at one point (one solution to the system) or the lines might be parallel and not intersect at all (no solutions to the system) or the lines could be completely co-linear (infinitely many solutions to the system). These are our only choices. We will see what kinds of systems cause these particular results as we solve systems of two linear equations.

Example 11.1.2.

Find the solution to the system: $\begin{cases} 5x - y = -1 \\ -x + y = -5 \end{cases}$

We rewrite the second equation to $y = x - 5$ and substitute it into the first equation:

$$5x - (x - 5) = -1$$

$$4x + 5 = -1$$
$$4x = -6$$
$$x = -\frac{6}{4} = -\frac{3}{2}$$

We substitute this value for x into the second equation:

$$-x + y = -5 \Longrightarrow -(-\frac{3}{2}) + y = -5 \Longrightarrow \frac{3}{2} + y = -\frac{10}{2} \Longrightarrow y = -\frac{13}{2}$$

Once we find a constant value for one variable, it does not matter which system equation is used to find the value of the second variable. We should get the same answer. A good stategy is to use one equation to *find* the value and the other equation to *check* the value. We check our answer for Example 11.1.2 in the other equation:

$$5x - y = 5(-\frac{3}{2}) - (-\frac{13}{2}) = -\frac{15}{2} + \frac{13}{2} = -\frac{2}{2} = -1. \ \surd$$

Two equations are the same if they have exactly the same solution set. For example, the equation $5x - y = -1$ is the same as $5x - y + 3 = 2$. We have simply added 3 to both sides of the equation. These equations are also equal to $3x - y = -1 - 2x$. Here we have subtracted $2x$ from both sides of $5x - y = -1$ [or subtracted $(2x + 3)$ from both sides of $5x - y + 3 = 2$]. If we add (subtract) the same thing to (from) both sides of an equation we don't change the solution set for that equation. We can also write equivalent equations by multiplying or dividing both sides of an equation by the same thing (except zero, of course). $5x - y = -1$ is the same as (has the same solution set as) $10x - 2y = -2$. We use this idea to help us solve systems of equations. Let's return to the system in Example 11.1.2: $\begin{cases} 5x - y = -1 \\ -x + y = -5 \end{cases}$

The solution to a system requires that both equations be true at the same time. We start with the first equation, $5x - y = -1$ and we rewrite it by adding the same thing to both sides. Since we know that in this system $-x + y$ is the same as -5, if we add the $-x + y$ to the left of our original equation and add -5 to the right of our original equation, then we haven't changed the solution set of the original equation: $5x - y = -1 \Longrightarrow (5x - y) + (-x + y) = -1 + -5 \Longrightarrow 4x = -6 \Longrightarrow x = -\frac{3}{2}$. So, the statement $x = -\frac{3}{2}$ is the direct result of assuming the simultaneous truth of the two equations in our system. We can now go back to either of the original statements in the system to get the y-value, $-\frac{13}{2}$. In doing so, we are really doing a series of the same operations on both sides of the equations. Ultimately, the system $\begin{cases} 5x - y = -1 \\ -x + y = -5 \end{cases}$ is equal to the system $\begin{cases} x = -\frac{3}{2} \\ y = -\frac{13}{2} \end{cases}$. Both have the same solution set.

The method just used to solve the system above is called the Method of Elimination. It is usually easier to work with when presented in a vertical orientation: $\begin{array}{rcrcr} 5x & - & y & = & -1 \\ -x & + & y & = & -5 \\ \hline 4x & & & = & -6 \end{array}$

We start with the first equation. Since $-x + y$ is equal to -5 we are adding the same thing to both sides of the first equation. We do our addition vertically and get the resulting equation $4x = -6$.

We then finish the problem as above.

You can probably now see why this is called the Method of Elimination.

We were fortunate in the above problem that our y terms were eliminated by simple addition. Life isn't always this easy. However, we can still use this method if instead of using both system equations as written we use an equivalent equation for one or both of the originals. We do not change the system solution if we replace one (or both) of the equations with equivalent equations. An equivalent equation can be made by multiplying both sides of an original equation by the same non-zero real number.

Suppose we wish to use the Method of Elimination on the system: $\begin{cases} 3x + 4y = -8 \\ -x - 3y = -4 \end{cases}$.

We can easily see that neither the x terms nor the y terms will be eliminated by simple vertical addition.

We can fix this difficulty by replacing the bottom equation with an equivalent equation gotten by multiplying both sides of the bottom equation by 3. We then add and eliminate the x term.

$$\begin{array}{rcrcr} 3x & + & 4y & = & -8 \\ -3x & - & 9y & = & -12 \\ \hline & - & 5y & = & -20 \end{array}$$

This tells us that $y = 4$. We substitute $y = 4$ into the first original equation to get $3x + 4(4) = -8 \Longrightarrow 3x = -24 \Longrightarrow x = -8$. We check the point $(-8, 4)$ in the second system equation, $-(-8) - 3(4) = 8 - 12 = -4$ $\surd$ Thus the only solution to the original system of linear equations is the point $(-8, 4)$. This tells us that we have two lines that intersect only once.

We could have chosen to solve the above system by eliminating the y term instead of the x term. In this case we avoid introducing fractions by rewriting BOTH of the original equations. We multiply the top one by 3 and the bottom one by 4. Look at the work below and figure out why we chose those particular values.

$$\begin{cases} 3x + 4y = -8 \\ -x - 3y = -4 \end{cases} \Longrightarrow \begin{array}{rcrcr} 9x & + & 12y & = & -24 \\ -4x & - & 12y & = & -16 \\ \hline 5x & & & = & -40 \end{array}$$

This last equation gives us $x = -8$. We can find $y = 4$ and check our answer as per usual.

Example 11.1.3.

Solve the following system three ways. Use the method of Substitution. Use the Method of Elimination twice, once eliminating the x term and once eliminating the y term.

$$\begin{cases} 3x - 2y = 4 \\ -5x + 4y = 2 \end{cases}$$

Substitution Method:

$3x - 2y = 4 \Longrightarrow -2y = 4 - 3x \Longrightarrow y = -2 + \frac{3}{2}x$.

We substitute this expression for y into the second equation:

$-5x + 4y = 2 \Longrightarrow -5x + 4(-2 + \frac{3}{2}x) = 2 \Longrightarrow -5x - 8 + 6x = 2 \Longrightarrow x = 10$. *We substitute $x = 10$ in the first equation to find y:* $3(10) - 2y = 4 \Longrightarrow 30 - 2y = 4 \Longrightarrow -2y = -26 \Longrightarrow y = 13$. *We check the point $(10, 13)$ in the second equation:* $-5(10) + 4(13) = -50 + 52 = 2$ $\surd$

Method of Elimination—eliminating x:
We multiply the first equation by 5 and the second equation by 3:

$$\begin{cases} 3x - 2y = 4 \\ -5x + 4y = 2 \end{cases} \Longrightarrow \begin{array}{rcrcr} 15x & - & 10y & = & 20 \\ -15x & + & 12y & = & 6 \\ \hline & & 2y & = & 26 \end{array}$$

This last line tells us that $y = 13$. We substitute into the first equation: $3x-2(13) = 4 \Longrightarrow 3x-26 = 4 \Longrightarrow 3x = 30 \Longrightarrow x = 10$. It isn't necessary to check this result since we did it above for the Substitution Method.

Method of Elimination—eliminating y:
We only need to multiply one equation. (Why?) We multiply the first equation by 2:

$$\begin{cases} 3x - 2y = 4 \\ -5x + 4y = 2 \end{cases} \Longrightarrow \begin{array}{rcrcr} 6x & - & 4y & = & 8 \\ -5x & + & 4y & = & 2 \\ \hline x & & & = & 10 \end{array}$$

We substitute into the first equation: $3(10) - 2y = 4 \Longrightarrow 30 - 2y = 4 \Longrightarrow -2y = -26 \Longrightarrow y = 13$.

Consider the system of equations $\begin{cases} 4x - 3y = 1 \\ -8x + 6y = -2 \end{cases}$

We attempt to solve using the Method of Elimination. If we multiply the top equation by 2 we get:

$$\begin{array}{rcrcr} 8x & - & 6y & = & 2 \\ -8x & + & 6y & = & -2 \\ \hline 0 & & & = & 0 \end{array}$$

We were certainly successful at eliminating things! If we look more closely at the two equations in the system we see that they are equivlent statements. If we multiply the top one by -2 we actually get the bottom one. The two equations are just different algebraic representations for the same line. So the solution sets for both equations are exactly the same. The solution set for the system, then is that same set too, the set of all order pairs that comprise the line. We say that the two equations in the system are co-linear. We will get this result when our attempts to solve the system result in a statement that is always true, such as $0 = 0$.

Let's make a slight change to the previous system and consider the system $\begin{cases} 4x - 3y = 1 \\ -8x + 6y = 3 \end{cases}$

Again we use the Method of Elimination. We multiply the top equation by 2 and get:

$$\begin{array}{rcrcr} 8x & - & 6y & = & 2 \\ -8x & + & 6y & = & 3 \\ \hline 0 & & & = & 5 \end{array}$$

It is no surprise that we again eliminated both of our variables. But our result is different. We ended up with the statement $0 = 5$. This is never true. This tells us that our two equations cannot both be true at the same time. The lines have no solutions in common. The lines do not intersect. It must be that the lines are parallel. What do we know about parallel lines? They have the same slope, but not the same y-intercept. If we rewrite the two equations into slope-intercept form our system looks like $\begin{cases} y = \frac{4}{3}x - \frac{1}{3} \\ y = \frac{4}{3}x + \frac{1}{2} \end{cases}$

In this form it is clear that both lines have slope $\frac{4}{3}$ but different y-intercepts. When we attempt to solve a system of parallel lines we will arrive at a statement that is always false, such as $0 = 5$.

Important Idea 11.1.1.

When we algebraically try to solve a system of equations and get a resulting statement that is always true, then the equations in the system are identical. The solution set for the system is the solution set for any one of the equations in the system. In the case of a system of linear equations, *'he system are co-linear.*

..gebraically try to solve a system of equations and get a resulting statement that is ..use, then the equations in the system have no solution. There is no common intersection ..ong the graphs of the equations in the system. In the case of a system of linear equations, the equations in the system represent parallel lines.

11.2 Systems of Inequalities

Graphing Inequalities

Suppose we want to represent the solution set for $y \geq 3$ on a set of Cartesian coordinates. We know how to graph the equation $y = 3$. It is a horizontal line with y-intercept 3. These points should be included in the solution set for $y \geq 3$. What other points are in the set? Certainly $(2,4)$, $(-7,9)$, and $(0,\pi)$ are points that satisfy the inequality $y \geq 3$. In fact any point in the plane that lies above the horizontal line $y = 3$ must be included. By similar reasoning, any point below the the line $y = 3$ should not be included. To graph our inequality then we draw the horizontal line $y = 3$ and then shade in the entire region above the line. The shaded region represents the solution set.

Now we want to graph $x < 2$ on a set of Cartesian coordinates. Again, we know how to graph $x = 2$. This is a vertical line with x-intercept 2. This time we do not want to include the points ON the line because we were given a strict inequality which doesn't include x being equal to 2. We do want to include all of the points in the plane that lie to the left of the line $x = 2$ and none of the points that lie to the right. We show this by drawing the boundary $x = 2$ as a dotted line and then shading the region to the left of this line. The idea of using a dotted line to indicate a boundary whose points are not included is similar to the idea of graphing on a number line where we use an open circle to indicate a boundary point that is not included in the desired set.

We can have more interesting graphs than just those bounded by horizontal or vertical lines. Let's show on a graph the solutions to the inequality $y > x + 2$. We know how to draw the line $y = x + 2$. This line is the boundary for our graph. Since we have a strict inequality we draw this line as a dotted line. We have to decide which side of the boundary includes our solution points. Since the inequality says that the y values are GREATER THAN the $x+2$ values we want to shade above the line. It is always a good idea to choose a test point in the region you believe you want to shade, just to make sure. This is not an insult....it is easy to make a sign mistake with the algebra of inequalities. A point that is definitely above the line is $(0, 100)$. Do we get a true statement when we substitute $x = 0$ and $y = 100$ into the original inequality? Yes. One point that is very often useful to use as a test point is the origin, the point $(0,0)$. If we substitute the point $(0,0)$ into the original inequality we get a false statement. This means that the region that includes the test point $(0,0)$ should NOT be shaded as part of the solution. We can feel pretty good about the accuracy of our graph now.

Example 11.2.1.
Graph the solution set for $x - 2y \leq 6$.

First we rewrite the inequality so that it will be in a format that is easier to graph.

$$x - 2y \leq 6 \Longrightarrow -2y \leq -x + 6 \Longrightarrow y \geq \frac{1}{2}x - 3.$$

Notice that we had to change the inequality sign when we divided both sides by -2 in the last step.

We now draw the line $y = \frac{1}{2}x - 3$. We make it a solid line to indicate that these values are included in the solution set. From the statement $y \geq \frac{1}{2}x - 3$ we understand that we want the points whose y values are greater than or equal to $\frac{1}{2}x - 3$, so we shade above the line. If we test the point $(0, 0)$ in the original problem we get a false statement. This confirms that the region containing the point $(0, 0)$ is not the desired region.

In the last example we had to change the inequality sign as part of our algebra. It is important that when we check our shaded region with a test point that we use the *original* equation for testing. Otherwise we could be making a test in an inequality where we might have made an algebraic error.

Example 11.2.2.
Sketch the graph of $x^2 + y^2 < 4$.

We know that the graph of $x^2 + y^2 = 4$ is a circle of radius 2 with center at the origin. This is our boundary. We draw a dotted line circle because we are given a strict inequality. We want the sum of the squares $x^2 + y^2$ to be less than 4 so we shade the interior of the circle. Using the test point $(0, 0)$ confirms this.

We can use test points to help us determine or verify which region of the plane needs to be shaded. But it would be good to not rely soley on this. THINK about what the inequality is saying and use reason rather than ritual to figure out the shading. In the last example we can see that we are adding two positive numbers, x^2 and y^2. If we want to keep their sum LESS THAN 4, then we can't use the outside of the circle for shading. The outside of the circle contains some very big values for x and y. The inequality that describes shading the outside of the circle is $x^2 + y^2 > 4$.

Example 11.2.3.
Sketch the graph of $y \leq x^2 - 1$.

We can easily sketch the graph of $y = x^2 - 1$. It is the standard parabola $y = x^2$ shifted down one unit. We draw this with a solid line because we do not have a strict inequality. We apply reason: We want in our set the points whose y values are LESS THAN OR EQUAL TO those of the parabola. So we shade below the parabola line. We shade everywhere that is not the "interior" of the parabola. If we test $(0, 0)$ we get a false statement, which helps verify our choice.

Systems of Inequalities

We know that a system of equations is a pair (or more) of equations and that the solution to a system of equations is the set of all ordered pairs (x, y) that are solutions to *all* of the equations in the system. We have a similar definition for a system of inequalities.

Given the system of inequalities $\begin{cases} y > 3 \\ x \leq 2 \end{cases}$ the solution of the system is the set of all points (x, y) that are in the solution set of both $y > 3$ and $x \leq 2$. We can draw the solution sets of each inequality and then look at their intersection. We saw at the beginning of the section that the graph for $y > 3$ consists of a dotted horizontal line through $(0, 3)$ and shading above the line. We saw that the solution for $y \leq 2$ is a solid vertical line through $(2, 0)$ with shading to the left of that line. The intersection of these two graphs is a shaded area bounded below by the dotted line $y = 3$ and bounded on the right by the vertical solid line $x = 2$. The intersection of the two boundary lines is the point $(2,3)$. We do not include the point $(2,3)$ in the solution set. (Why?)

The intersections of boundaries for the graph of a system of inequalites are called *vertices*. It is important to specifically state the vertices of the solution so that the boundaries are unambiguous. State whether or not the vertices are part of the solution set. This makes it very clear what is in and what is not in the solution set. There are many applications in economics and optimization theory where the identification of these vertices is very important.

It will be very helpful to your understanding of the next examples if you sketch the graphs as you work through them.

Example 11.2.4.

Find the solution to the system $\begin{cases} x - 2y \le 6 \\ x > -1 \end{cases}$

We rewrite the first inequality (from Example 11.2.1) into the form $y \ge \frac{1}{2}x - 3$. *We sketch the graph of the line* $y = \frac{1}{2}x - 3$, *making it a solid line. We shade above the line.*
We sketch the graph of $x > -1$: *We draw a dotted vertical line through the point* $(-1,0)$ *and we shade to the right of this dotted line.*
The intersection of these two graphs is the solution set. It is a triangular shaped region bounded on the bottom by the solid line $y = \frac{1}{2}x - 3$ *and bounded on the left by the vertical dotted line* $x = -1$.
The vertex of the region is the intersection of the two boundary lines $(-1, -\frac{7}{2})$. *The vertex is not included in the solution.*

Example 11.2.5.

Find the solution to the system $\begin{cases} x - 2y \le 6 \\ x > -1 \\ y \le 2 \end{cases}$

We notice that the first two inequalities are the same as in Example 11.2.4, so we can begin with the solution graph from that example.
We now add the graph of $y = 2$, *a horizontal line through the point* $(0,2)$. *We make this a solid line and shade below it.*
The intersection of the three solution sets (the solution of the first two now intersected with the solution set of the third) is a triangle. It is bounded on the left by dotted line $x = -1$, *on the top by solid line* $y = 2$ *and on the bottom by solid line* $y = \frac{1}{2}x - 3$. *The vertices are* $(-1,2)$ *(not in the solution set),* $(-1, -\frac{7}{2})$ *(not in the solution set) and* $(10,2)$ *(in the solution set).*

In the last example, the boundaries for the solution formed a triangle. It was clear that we had three vertices. The point $(-1,2)$ was obvious from the graph. The other two points had to be calculated by finding the intersections of the respective pairs of lines. The intersection of lines $x = -1$ and $y = \frac{1}{x} - 3$ is $(-1, -\frac{7}{2})$. The intersection of the lines $y = 2$ and $y = \frac{1}{2}x - 3$ is $(10,2)$. Finding intersections is a matter of solving each pair of equations simultaneously.

Example 11.2.6.

Find the solution to the system $\begin{cases} 3x \ge y + 5 \\ x - 3y < -1 \end{cases}$

We rewrite the two linear inequalities so that they are easier to graph:

$$3x \ge y + 5 \Longrightarrow 3x - 5 \ge y \Longrightarrow y \le 3x - 5$$

$$x - 3y < -1 \Longrightarrow -3y < -x - 1 \Longrightarrow y > \frac{1}{3}x + \frac{1}{3}$$

We graph the line $y = 3x - 5$ with a solid line and shade the region below (to the right) of it.
We graph the line $y = \frac{1}{3}x + \frac{1}{3}$ with a dotted line and shade the region above it.
The intersection of these two graphs is the triangular shaped region in the upper right portion of the axes. It is bounded below by the dotted line $y = \frac{1}{3}x + \frac{1}{3}$ and bounded on the left by the solid line $y = 3x - 5$. There is one vertex point, $(2, 1)$ which is not part of the solution set.

Example 11.2.7.

Find the solution to the system $\begin{cases} y > x^2 - 2 \\ x + y > 0 \end{cases}$

We rewrite the second equation as $y > -x$.
The graph of the first equation is the parabola $y = x^2$ shifted down two units. We graph this with a dotted line. Since we have y is greater than $x^2 - 2$ we shade the interior of the parabola.
Now we graph the line $y = -x$, using a dotted line. We shade above the line.
The intersection of these two graphs is not the little "banana-shaped" piece enclosed by the two graphs. The intersection is the interior of the parabola that is above the "banana." The solution set is bounded on the left and right by the dotted line parabola and is bounded on the bottom by the dotted line $y = -x$. There are two vertices. To find their coordinates we find where the dotted lines intersect by solving the system $\begin{cases} y = x^2 - 2 \\ y = -x \end{cases}$ *We can use the substitution method, substituting for y:*

$$y = x^2 - 2 \Longrightarrow -x = x^2 - 2 \Longrightarrow 0 = x^2 + x - 2$$
$$\Longrightarrow 0 = (x + 2)(x - 1) \Longrightarrow x = -2 \text{ or } x = 1$$
$$x = -2 \Rightarrow y = 2 \text{ and } x = 1 \Rightarrow y = -1$$

The two vertices are $(-2, 2)$ and $(1, -1)$. Neither is part of the solution set.

Example 11.2.8.

Find the solution to the system $\begin{cases} x^2 + y^2 < 4 \\ y > x \end{cases}$

The graph of the equation $x^2 + y^2 = 4$ is a circle with center at the origin and radius 2. We draw that with a dotted line. Since $x^2 + y^2$ is less than 4 we shade the interior of the circle.
The graph of $y = x$ is a line through the origin. We include this dotted line on the graph and shade above it.
The intersection of these two graphs is the interior of a semicircle that sits in quadrant II and in half of each of quadrants I and III. There are two vertices where the line $y = x$ intersects the circle $x^2 + y^2 = 4$. To find the coordinates we can formally solve the system $\begin{cases} x^2 + y^2 = 4 \\ y = x \end{cases}$ *or just recognize that since these coordinates are of the form (x, x), the points must be $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$. Neither vertex is part of the solution set.*

Example 11.2.9.

Find the solution to the system $\begin{cases} x \geq y^2 - 1 \\ xy > 0 \end{cases}$

The graph of $x = y^2 - 1$ is a horizontal parabola with vertex $(-1, 0)$, opening to the right. We graph this with a solid line. We are interested in those points whose x values are greater than $y^2 - 1$ so we shade the interior of the parabola. (If you are unsure what to shade, don't forget that you can pick a test point).
The inequality $xy > 0$ is interesting because at first it looks impossible to deal with algebraically. But, think: we have two values, x and y, whose product is positive. What does that tell us about x and y? They both must have the same sign. This is true for all points in the first and third quadrants.

This then is the solution set for the inequality $xy > 0$. Shade all of the first and third quadrants. The intersection of the solution sets for these two inequalities has two pieces. One piece is enclosed in the third quadrant, bounded below by the solid line parabola $x = y^2 - 1$ and bounded by the dotted line x and y axes. There are three vertices for this piece. We can readily see from the graph that their coordinates are $(0,0)$, $(-1,0)$ and $(0,-1)$. None are part of the solution set. The second piece of the solution set is entirely in the first quadrant. It is bounded above by the solid line parabola $x = y^2 - 1$, bounded below by the dotted line x-axis and bounded on the left by the dotted line y-axis. The two vertices of this piece are $(0,0)$ and $(0,1)$. Neither is a part of the solution.

Example 11.2.10.

Find the solution to the system $\begin{cases} x + y \le 3 \\ 2x - y \le 7 \\ x + 3y > -6 \end{cases}$

We recognize that each of these inequalities is linear and rewrite the system so that it is easier to work with:

$$\begin{cases} y \le 3 - x \\ y \ge 2x - 7 \\ y > -\frac{1}{3}x - 2 \end{cases}$$

When dealing with a system of more than two inequalties it can be helpful to label each inequality line on the graph. We will call our inequalities T, M, B (top, middle, bottom — referring to their list position in the system). We graph the line $y = 3 - x$ with a solid line, label the line T and shade below it. We graph line $y = 2x - 7$ with a solid line, label it M, and shade above (to the left of) it. We graph line $y = -\frac{1}{3}x - 2$ with a dotted line, label it B and shade above it. What is the intersection of these three graphs? It is not the enclosed triangle.

The solution to the system is the region bounded above by the solid line T, bounded below by the dotted line B and bounded on the right by the solid line M. There are two vertices. We need to find the intersections of lines T and M and of lines M and B. We do not care about the intersection of lines T and B. It is not relevant to the solution.

To get the intersection of lines T and M we solve the system $\begin{cases} y = 3 - x \\ y = 2x - 7 \end{cases}$

We get point $(\frac{10}{3}, -\frac{1}{3})$. This vertex is part of the solution set.

To get the intersection of lines M and B we solve the system $\begin{cases} y = 2x - 7 \\ y = -\frac{1}{3}x - 2 \end{cases}$

We get the vertex point $(\frac{15}{7}, -\frac{19}{7})$. This point is not part of the solution set.

11.3 Exercises

Problems for Section 11.1

Problem 1. Solve the following systems of equations.

(a) $\begin{cases} y = x + 3 \\ y = 9 - x^2 \end{cases}$ (b) $\begin{cases} xy = 4 \\ y = 4x \end{cases}$ (c) $\begin{cases} y = |\sin x| & \text{for } x \in [0, 2\pi] \\ y = |\cos x| & \text{for } x \in [0, 2\pi] \end{cases}$

(d) $\begin{cases} y = x^2 \\ y = -x^2 + 2 \\ y = 2x - 1 \end{cases}$ (e) $\begin{cases} \log x + \log y = 2 \\ \log(2x) - \log y = 1 \end{cases}$ (f) $\begin{cases} y = \sin(2x) & \text{for } x \in [0, 2\pi] \\ y = \cos x & \text{for } x \in [0, 2\pi] \end{cases}$

Problem 2. Graph the equations in the following systems to determine how many solutions there are and to get an estimate of what they are. Then algebraically solve the system to verify your graphing results.

(a) $\begin{cases} x^2+y^2=4 \\ y=-x+2 \end{cases}$ (b) $\begin{cases} x^2-y^2=4 \\ 3x-2y=0 \end{cases}$ (c) $\begin{cases} y=\sin(2x) & \textit{for } x\in[0,\pi] \\ y=\sin x & \textit{for } x\in[0,\pi] \end{cases}$

Problem 3. Graph the equations in the following systems to determine how many solutions there are. You do not need to find actual solution values.

(a) $\begin{cases} y=\frac{1}{\pi}x \\ y=\lvert\cos x\rvert \end{cases}$ (b) $\begin{cases} y=\lvert x\rvert \\ y=e^x \end{cases}$

Problem 4. Use the method of substitution to solve the following systems of linear equations. Indicate if you found parallel lines or co-linear equations.

(a) $\begin{cases} 3x-y=-3 \\ 3y+19=-5x \end{cases}$ (b) $\begin{cases} 2x-5y=-7 \\ 4y-2x=10 \end{cases}$

Problem 5. Use the method of elimination to solve the following systems of linear equations. Indicate if you found parallel lines or co-linear equations.

(a) $\begin{cases} x+2y=6 \\ 2x-y=7 \end{cases}$ (b) $\begin{cases} 3x-2y=-19 \\ x+4y=-4 \end{cases}$

(c) $\begin{cases} 3x-2y=12 \\ 5x+4=2y \end{cases}$ (d) $\begin{cases} 3x+2y=-5 \\ 4x+3y=1 \end{cases}$

Problem 6. Use any method you wish to solve the following systems of linear equations. Indicate if you found parallel lines or co-linear equations.

(a) $\begin{cases} -8x+y=-2 \\ 4x-3y=1 \end{cases}$ (b) $\begin{cases} 3y=2-x \\ 2x+6y=-3 \end{cases}$ (c) $\begin{cases} y=3x-3 \\ 6x=8+3y \end{cases}$

(d) $\begin{cases} \frac{2}{3}x+\frac{3}{4}y=\frac{5}{6} \\ \frac{4}{3}x-\frac{1}{4}y=\frac{1}{2} \end{cases}$ (e) $\begin{cases} \frac{1}{2}x-y=-3 \\ -x+2y=6 \end{cases}$

Problems for Section 11.2

Problem 1. Graph the following inequalities on Cartesian coordinate axes.

(a) $y>4$ (b) $x\le -\frac{1}{2}$ (c) $y<2x-5$ (d) $3x-y\le 5$

(e) $x\ge y^2$ (f) $\dfrac{x^2}{4}+\dfrac{y^2}{9}\ge 1$ (g) $\lvert x\rvert>3$

Problem 2. **Graph the solution to each system of inequalities. Find the coordinates of all vertices of your solution and tell whether they are included in the solution.**

(a) $\begin{cases} y < 4 \\ y > x \end{cases}$ (b) $\begin{cases} y > x + 2 \\ y < \frac{1}{2}x + 2 \end{cases}$ (c) $\begin{cases} |y| < 1 \\ |x| < 2 \end{cases}$

(d) $\begin{cases} x + y > 2 \\ x < 3 \\ y < 5 \end{cases}$ (e) $\begin{cases} y < x + 3 \\ -2y < 3x - 6 \\ y > \frac{1}{2}x - 1 \end{cases}$ (f) $\begin{cases} y < -x^2 + 3 \\ y > x^2 - 1 \end{cases}$

(g) $\begin{cases} y < x^2 \\ y \geq -x^2 \end{cases}$ (h) $\begin{cases} y < -x^2 + 4 \\ y \leq 2x + 1 \end{cases}$ (i) $\begin{cases} y \geq x^2 \\ x \geq y^2 \end{cases}$

(j) $\begin{cases} y > x^2 - 1 \\ xy \leq 0 \end{cases}$ (k) $\begin{cases} y < \sin x \\ y > -\sin x \end{cases}$ (l) $\begin{cases} y < e^x \\ y > \ln x \end{cases}$

(m) $\begin{cases} \dfrac{x^2}{4} + \dfrac{y^2}{9} < 1 \\ \dfrac{x^2}{9} + \dfrac{y^2}{4} > 1 \end{cases}$

Problem 3. **On a set of coordinate axes, draw a circle with center at the origin and radius 3. On the same set of axes, draw a dotted-line circle with center at the origin and radius 1. Shade the donut shaped region between the two graphs. Write a system of inequalities to algebraically describe your sketch.**

Problem 4. **On a set of coordinate axes, plot the points $(0,3)$, $(6,0)$ and $(-2,-1)$. Connect the points with dotted line segments to form a dotted triangle. Shade the interior of the triangle. Write a system of inequalities that describes your shaded region.**

11.4 Answers to Exercises

Answers for Section 11.1 Exercises

Answer to Problem 1.

(a) $(-3,0)$, $(2,5)$ (b) $(1,4)$, $(-1,-4)$ (c) $(\frac{\pi}{4}, \frac{\sqrt{2}}{2})$, $(\frac{3\pi}{4}, \frac{\sqrt{2}}{2})$, $(\frac{5\pi}{4}, \frac{\sqrt{2}}{2})$, $(\frac{7\pi}{4}, \frac{\sqrt{2}}{2})$

(d) $(1,1)$ (e) $(10\sqrt{5}, 2\sqrt{5})$ (f) $(\frac{\pi}{6}, \frac{\sqrt{3}}{2})$, $(\frac{\pi}{2}, 0)$, $(\frac{5\pi}{6}, -\frac{\sqrt{3}}{2})$, $(\frac{3\pi}{2}, 0)$

Answer to Problem 2.

(a) $(0,2)$, $(2,0)$ (b) no solution (c) $(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$, $(0,0)$, $(\pi,0)$

Answer to Problem 3.

(a) three solutions (b) one solution

Answer to Problem 4.

(a) $(-2,-3)$ (b) $(-11,-3)$

Answer to Problem 5.
(a) $(4, 1)$ (b) $(-6, \frac{1}{2})$ (c) $(-8, -18)$ (d) $(-17, 23)$

Answer to Problem 6.
(a) $(\frac{1}{4}, 0)$ (b) No solution; parallel lines (c) $(\frac{1}{3}, -2)$ (d) $(\frac{1}{2}, \frac{2}{3})$ (e) colinear

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Answers for Section 11.2 Exercises

Answer to Problem 1.
Graphs not shown.

Answer to Problem 2.
(a) $(4, 4)$ no (b) $(0, 2)$ no (c) $(-2, 1)$ no, $(2, 1)$ no, $(2, -1)$ no, $(-2, -1)$ no

(d) $(3, 5)$ no, $(-3, 5)$ no, $(3, -1)$ no (e) $(0, 3)$ no, $(2, 0)$ no (f) $(\sqrt{2}, 1)$ no, $(-\sqrt{2}, 1)$ no

(g) $(0, 0)$ no (h) $(1, 3)$ no, $(-3, -5)$ no (i) $(0, 0)$ yes, $(1, 1)$ yes

(j) $(0, 0)$ yes, $(1, 0)$ no, $(-1, 0)$ no, $(0, -1)$ no (k) $(n\pi, 0)$ no [for n an integer]

(l) no vertices (m) $(\frac{6}{\sqrt{13}}, \frac{6}{\sqrt{13}})$ no, $(-\frac{6}{\sqrt{13}}, \frac{6}{\sqrt{13}})$ no, $(\frac{6}{\sqrt{13}}, -\frac{6}{\sqrt{13}})$ no, $(-\frac{6}{\sqrt{13}}, -\frac{6}{\sqrt{13}})$ no

Answer to Problem 3.

$$\begin{cases} x^2 + y^2 \le 9 \\ x^2 + y^2 > 1 \end{cases}$$

Answer to Problem 4.

$$\begin{cases} y < 2x + 3 \\ y < -\frac{1}{2}x + 3 \\ y > \frac{1}{8}x - \frac{3}{4} \end{cases}$$