CORRECTION TO *HOMOTOPY GROUPS OF THE COMBINATORIAL GRASSMANNIAN*

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This is a correction of the proof of Proposition 3.1 in [**?**]. Thanks to Nevena Palic for pointing out the error.

The original proof was broken into three assertions, of which two were correct:

(1) the closure $\overline{U_M}$ equals $\bigcup_{M' \in \mathrm{MacP}(k,n)_{\leq M}} U_{M'}$.
(2) U_M is contractible.

(We note one small correction to their proofs: the assumption that 1 and 2 were nonloops should be an assumption that $\{1,2\}$ is a basis for M.)

The third assertion, that $\overline{U_M}$ is a topological ball, is not so clear. We will give an alternative proof of the proposition that sidesteps the issue of whether the assertion is true or not.

Lemma 0.1. *If $M_0, \ldots, M_m \in \mathrm{MacP}(2,n)$ and $M_m \rightsquigarrow \cdots \rightsquigarrow M_0$ then $\bigcup_{i=0}^m U_{M_i}$ is contractible.*

Proof. We will show that $\bigcup_{i=0}^m U_{M_i}$ deformation retracts to $\bigcup_{i=0}^{m-1} U_{M_i}$.

As in the original proof we assume without loss of generality that $\{1,2\}$ is a basis for M_0, and hence for each M_i. Thus each element of $\bigcup_{i=0}^m U_{M_i}$ has a unique expression as the row space of a matrix $\begin{pmatrix} 1 & 0 & v_{1,3} & \cdots & v_{1,n} \\ 0 & 1 & v_{2,3} & \cdots & v_{2,n} \end{pmatrix}$. Let $\mathbf{v}_i$ denote the ith column of this matrix. Thus the above matrix will be denoted $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$.

If $e \in [n]$ and $M'_0, \ldots, M'_m$ are obtained from $M_0, \ldots, M_m$ by reorienting e then $M'_m \rightsquigarrow \cdots \rightsquigarrow M'_0$ and $\bigcup_{i=0}^m U_{M_i} \cong \bigcup_{i=0}^m U_{M'_i}$. Thus by reorienting if necessary we may assume that no M_i contains a pair of antiparallel nonloops, i.e., no M_i has a circuit of the form $\{e,f\}^+$.

Choose a representative e_P from each parallllelism class P of nonloops in M_{m-1} in such a way that 1 and 2 are the representatives for their parallelism classes. Define a homotopy

$$H : \bigcup_{i=0}^m U_{M_i} \times I \to \bigcup_{i=0}^m U_{M_i}$$
$$(\mathrm{Row}(\mathbf{v}_1, \cdots, \mathbf{v}_n), t) \to \mathrm{Row}(\mathbf{w}_1(t), \cdots, \mathbf{w}_n(t))$$

where

$$\mathbf{w}_e(t) = \begin{cases} (1-t)\mathbf{v}_e & \text{if } e \text{ is a loop in } M_{m-1} \\ (1-t)\mathbf{v}_e + t\frac{\|\mathbf{v}_e\|}{\|\mathbf{v}_{e_P}\|}\mathbf{v}_{e_P} & \text{if } e \in P. \end{cases}$$

To see that H is well-defined and is our desired deformation retract, notice:

(1) If $\mathbf{v}_e$ is parallel to $\mathbf{v}_{e_P}$ for some P then $\mathbf{w}_e(t) = \mathbf{v}_e$ for all t. In particular, $\mathbf{w}_1(t) = \mathbf{v}_1$ and $\mathbf{w}_2(t) = \mathbf{v}_2$ for all t, and $H(V,t) = V$ for all $V \in \bigcup_{i=0}^{m-1} U_{M_i}$.

(2) Consider when $V \in U_{M_m}$. If e is in a parallelism class P of M_{m-1}, f is in a parallelism class Q of M_{m-1}, and $\mathbf{v}_f$ is clockwise from $\mathbf{v}_e$, then either $P = Q$ or each of $\{\mathbf{v}_f, \mathbf{v}_{e_Q}\}$ is clockwise from each of $\{\mathbf{v}_e, \mathbf{v}_{e_P}\}$. Since $\mathbf{w}_f$ is a positive linear combination of $\{\mathbf{v}_f, \mathbf{v}_{e_Q}\}$ and $\mathbf{w}_e$ is a positive linear combination of $\{\mathbf{v}_e, \mathbf{v}_{e_P}\}$, we see $\mathbf{w}_f(t)$ is clockwise from $\mathbf{w}_e(t)$ for all t if $P \neq Q$ and for all $t < 1$ if $P = Q$. Similarly, if one or both of e and f is a loop in M_{m-1} and $\mathbf{v}_f$ is clockwise from $\mathbf{v}_e$ then $\mathbf{w}_f(t)$ is clockwise from $\mathbf{w}_e(t)$ for all $t < 1$. Thus $H(V, t) \in U_{M_m}$ if $t < 1$ and $H(V, 1) \in U_{M_{m-1}}$.

□

Lemma 0.2. *If $M_0, \ldots, M_m \in \mathrm{MacP}^{(k+2)}(k, n)$ and $M_m \rightsquigarrow \cdots \rightsquigarrow M_0$ then $\bigcup_{i=0}^m U_{M_i}$ is contractible.*

Proof. Without loss of generality assume that $k+3, \ldots, n$ are loops in M_m (and therefore in each M_i), and let $M'_0, \ldots, M'_m \in \mathrm{MacP}(k, k+2)$ be obtained from $M_1, \ldots, M_m$ by deleting $\{k+3, \ldots, n\}$. Then $\bigcup_{i=0}^m U_{M_i} \cong \bigcup_{i=0}^m U_{M'_i}$. Further the homeomorphism $V \to V^\perp$ from $G(k, k+2)$ to $G(2, k+2)$ sends $\bigcup_{i=0}^m U_{M'_i}$ to $\bigcup_{i=0}^m U_{(M'_i)^*}$, and $(M'_m)^* \rightsquigarrow \cdots \rightsquigarrow (M'_0)^*$. Thus by Lemma **??** $\bigcup_{i=0}^m U_{M_i}$ is contractible. □

Revised proof of Proposition 3.1. We construct a homotopy inverse

$$f : \Delta\mathrm{MacP}^{(k+2)}(k, n) \to G^{(k+2)}(k, \mathbb{R}^n)$$

so that for each simplex σ of $\Delta\mathrm{MacP}^{(k+2)}(k, n)$ with vertex set $\{M_0, \ldots, M_m\}$, we have $f(\sigma) \subseteq \bigcup_{i=0}^m U_{M_i}$. For each $M \in \mathrm{MacP}(k, n)$ choose an $f(M) \in U_M$. Then once f is defined on the $(m-1)$-skeleton of $\Delta\mathrm{MacP}^{(k+2)}(k, n)$, for every m-simplex σ, f maps the boundary of σ into $\bigcup_{i=0}^m U_{M_i}$. By Lemma **??** this is contractible, so we can extend f to σ.

The composition $c \circ f : \Delta\mathrm{MacP}^{(k+2)}(k, n) \to \Delta\mathrm{MacP}^{(k+2)}(k, n)$ send every simplex to itself, hence is homotopic to the identity. We can construct a homotopy H from $f \circ c$ to the identity by taking the cell structure on $G^{(k+2)}(k, \mathbb{R}^n)$ with maximal cells $\tau \times I$ for each maximal simplex τ in our given triangulation of $G^{(k+2)}(k, \mathbb{R}^n)$ and inducting on skeleta. We define $H(x, 0) = f \circ c(x)$ and $H(x, 1) = x$ for all x, and thus for each cell σ in $G^{(k+2)}(k, \mathbb{R}^n)$ with vertex set $\{M_0, \ldots, M_m\}$ $H(\sigma \times \{0, 1\}) \subseteq \bigcup_{i=0}^m U_{M_i}$. Then as in our previous argument we can extend H to cells $\sigma \times I$ inductively.

References

[1] L. Anderson, Homotopy groups of the combinatorial Grassmannian. *Discrete and Computational Geometry*, **20** (1998), 549-560.