

Optimal Asset Allocation with Asymptotic Criteria

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Abstract

Assume (1) asset returns follow a stochastic multi-factor process with time-varying conditional expectations; (2) investments are linear functions of factors. This paper calculates asymptotic joint moments of the logarithm of investor's wealth and the factors. These formulas enable fast computation of a wide range of investment criteria. The results are illustrated by a numerical example that shows that the optimal portfolio rules are sensitive to the specification of the investment criterion.

1 Introduction

The asset returns are predictable: see (Cochrane 1999) and (Campbell, Lo & MacKinlay 1997) who overview the empirical support for asset return pre-

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dictability¹, and (Brennan, Schwartz & Lagnado 1997), (Campbell & Viceira 1999), and (Bielecki & Pliska 1999) who study optimal portfolio allocation in the situation with predictable returns. Conventional dynamic stochastic programming needs a lot of computing power to find optimal allocations, and a possible approach to this problem is to choose an approximate investment criterion that would make the problem numerically tractable. This paper studies portfolio optimization for a class of such criteria and tests the sensitivity of optimal portfolio rules to the choice of the investment criterion.

In early 1970s Samuelson and Merton derived Hamilton-Jacobi-Bellman equation for the solution of the dynamic asset allocation problem, an equation that is valid for any type of return process and any type of investment criteria.² This partial differential equation is, however, hard to solve. To circumvent this difficulty, (Bielecki & Pliska 1999)³ introduced the risk-sensitive investment criterion, which makes the optimization problem easier. The criterion assumes that the investor's preferences depend only on the growth rate of the expected logarithm of portfolio value and on the growth rate of its variance. This assumption appears to be excessively tight since it ignores other relevant moments, such as the covariance with the factors. An investor might be concerned with the performance of the portfolio in "bad" times, when his other sources of income

¹For example, see empirical studies by (Balvers, Cosimano & McDonald 1990), (Breen, Glosten & Jagannathan 1990), (Campbell 1987), (Cochrane 1991), (Fama & French 1989), (Pesaran & Timmermann 1995), (Pesaran & Timmermann 2000), (Cooper 1999).

²(Samuelson 1969), (Merton 1969), (Merton 1971), and (Merton 1973)

³See also (Stettner 1999), (Bielecki, Pliska & Sherris 2000), (Bielecki, Hernandez-Hernandez & Pliska 1999), (Bielecki & Pliska 2001)

are low. The criterion that depends only on variance and expectation of the portfolio value ignores this consideration.

This paper uses a new method to compute the asymptotic joint moments of the logarithm of the portfolio value and factors, allowing for the fast computation of a wide class of investment criteria. An example calibrated on the real data shows that the optimal portfolio rules strongly depend on the specification of the investment criterion. In particular, the rule is sensitive to the inclusion of the correlation of portfolio value with factors in the criterion. This supports the view that the correct choice of the approximate investment criterion is crucial for optimal portfolio allocation.

The remainder is organized as follows. Section 2 explains the model. Section 3 formulates the main result. Section 4 specializes the result to a one-factor example, calibrates it on the real data and presents numerical illustrations. And Section 5 concludes. The proof of the main result is in Appendix.

2 Model

Assumption 1. Securities follow a dynamic factor model.

$$\frac{dS_i(t)}{S_i(t)} = (a + AX(t))_i dt + \sum_{k=1}^{m+n} \sigma_{ik} dW_k(t), \quad i = 1, 2, \dots, m, \quad (1)$$

$$dX(t) = BX(t)dt + \Lambda dW(t), \quad (2)$$

where $W(t)$ is a \mathfrak{R}^{m+n} valued standard Brownian motion process, $X(t)$ is \mathfrak{R}^n valued factor process. It is assumed that $n \times n$ matrix B is stable. This coincides

with the model of (Bielecki & Pliska 1999) and (Bielecki et al. 2000) with the exception that the factor process $X(t)$ is normalized to have zero mean.

The process for investor's wealth U is

$$dU = \frac{IU}{S} dS, \quad (3)$$

where I is the investment vector measured in shares of wealth.

Assumption 2. I is a linear function of factors.

$$I = h + HX, \quad (4)$$

where H is a constant $m \times n$ matrix and h is a constant $m \times 1$ vector.⁴

Thus, the investor's wealth U follows the process

$$dU = U(h + HX)' \frac{dS}{S} \quad (5)$$

Assumption 3. The investment criterion is

$$W(h, H) = \liminf_{t \rightarrow \infty} \frac{1}{t} \{Eu(t) - \frac{\theta}{4} \text{var}(u(t))\} + \Gamma \lim_{t \rightarrow \infty} E(u(t)X(t)), \quad (6)$$

where $u(t) =: \ln U(t) - \ln U(0)$.

Note that this criterion includes the covariance of the logarithm of portfolio value with the factors. An interpretation of this criterion is that the investor cares not only about growth of the portfolio value and its volatility but also about the ability of the portfolio to generate good returns in bad times when his other income sources bring low returns.

⁴This is not a very stringent assumption because non-linear functions of the factors can always be added to the set of all factors to incorporate non-linear dependence of investments on the factors.

3 Result

The following theorem shows how to compute asymptotic moments that enter the investment criterion.

Theorem 1 *If assumptions 1 and 2 hold then*

$$E(u) = t[h'a - \frac{1}{2}h'\Sigma\Sigma'h + \text{tr}(\Delta(H'A - \frac{1}{2}H'\Sigma\Sigma'H))], \quad (7)$$

$$\text{Var}(u) \sim t[YY' + \text{tr}(2SH'A + (\Delta - S)H'\Sigma\Sigma'H)] + \text{const as } t \rightarrow \infty, \quad (8)$$

$$E(uX) \sim B^{-1}[\Delta(H'\Sigma\Sigma'h - A'h - H'a) - \Lambda\Sigma'h] \text{ as } t \rightarrow \infty. \quad (9)$$

Here $\Delta_{ij} =: E(x_i x_j)$, vector Y is defined as follows:

$$Y =: (h'\Sigma\Sigma'H - h'A - a'H)B^{-1}\Lambda + h'\Sigma, \quad (10)$$

and matrix S is the symmetric solution of the equation

$$BS + SB' = -2\Delta H'A\Delta + \Delta H'\Sigma\Sigma'H\Delta - 2\Lambda\Sigma'H\Delta. \quad (11)$$

The proof of Theorem 1 is in the Appendix. Let me explain here its main idea for the case with only one factor x . Suppose that we want to compute $\lim_{t \rightarrow \infty} E(u^i x^j)$ and we already know $\lim_{t \rightarrow \infty} E(u^k x^l)$ for $k < i$ and for $k = i, l < j$. Using Ito Lemma, we can write

$$\begin{aligned} d(u^i x^j) &= i u^{i-1} x^j du + j x^{j-1} u^i dx \\ &+ \frac{1}{2} \{ i(i-1) u^{i-2} x^j (du)^2 + j(j-1) u^i x^{j-2} (dx)^2 + ij u^{i-1} x^{j-1} du dx \}. \end{aligned} \quad (12)$$

After the expressions for du and dx are substituted and the expectations are taken, this equation is reduced to the following differential equation on $E(u^i x^j)$:

$$\frac{d}{dt}E(u^i x^j) = jBE(u^i x^j) + \dots, \quad (13)$$

where \dots includes only terms with already known asymptotic limits. Since B is stable (in this one factor example it means $B < 0$), the solution of this differential equation has an asymptotic limit that can be easily calculated. Theorem 1 extends this method to the multi-factor case.

4 Application

This section applies the general result to the particular example with the parameters estimated from real data. First, Theorem 1 is specialized to the case with only one asset and one factor. Thus, a , A , B , h , and H are scalars, $\Sigma = (\sigma, \eta)$, and $\Lambda = (0, \lambda)$.

In this setting, Theorem 1 is reduced to the following

Corollary 1

$$E(u) = t\left\{ha - \frac{\lambda^2}{2B}HA - \frac{\sigma^2 + \eta^2}{2}\left(h^2 - \frac{\lambda^2}{2B}H^2\right)\right\} \quad (14)$$

$$E(ux) \sim \frac{\lambda^2}{2B^2}\{hH(\eta^2 + \sigma^2) - Ah - Ha\} - \frac{\lambda\eta}{B}h \quad (15)$$

$$Var(u) \sim t\{YY' + 2SHA + \left(\frac{\lambda^2}{2B} - S\right)H^2(\sigma^2 + \eta^2)\} \quad (16)$$

where

$$Y = \{hH(\sigma^2 + \eta^2) - hA - aH\}\frac{1}{B}(0, \lambda) + h(\sigma, \eta)$$

$$S = \frac{\lambda^2}{4B^2}\left\{-2HA\frac{\lambda^2}{2B} + \frac{\lambda^2}{2B}H^2(\sigma^2 + \eta^2) - 2H\lambda\eta\right\}.$$

I have estimated the parameters of the stochastic process using the monthly data from CRSP files. The effective stock return is taken as the difference of the rate of return on S&P composite index and the short-term interest rate. The factor is the short-term interest rate, which is the rate on 3-month Treasury Bill. The results of estimation are summarized in the next two tables.

Table 1. Process Estimates^a

	constant	r
dS/S	1.993 (3.505)	-1.177 (-14.220)
dr	0.120 (0.911)	0.979 (42.885)

^aPeriod is from January 1970 to December 2000. Rates and returns are measured in percentage terms; t-ratios are in parentheses

Table 2. Covariances of interest rate and stock return innovations^a

	Stock Return	r
Stock Return	19.587	0.0553
r	0.0553	0.4006

^aPeriod is from January 1970 to December 2000

Therefore, the parameters of the model can be calibrated as $a = 0.01993$, $A = -0.01177$, $B = -0.021$, $\lambda = 0.6329$, $\eta = 0.000874$, and $\sigma = 0.044249$.

For the first illustration, assume that $h = 1$, so that on average the investor have all his funds in the stock. If the interest rate deviates from its average value then the investor can move his assets to bonds. He can also borrow additional funds and increase his position in the stock. The first three graphs show how

Eu , $Var(u)$, and $E(ux)$ change with respect to changes in H .

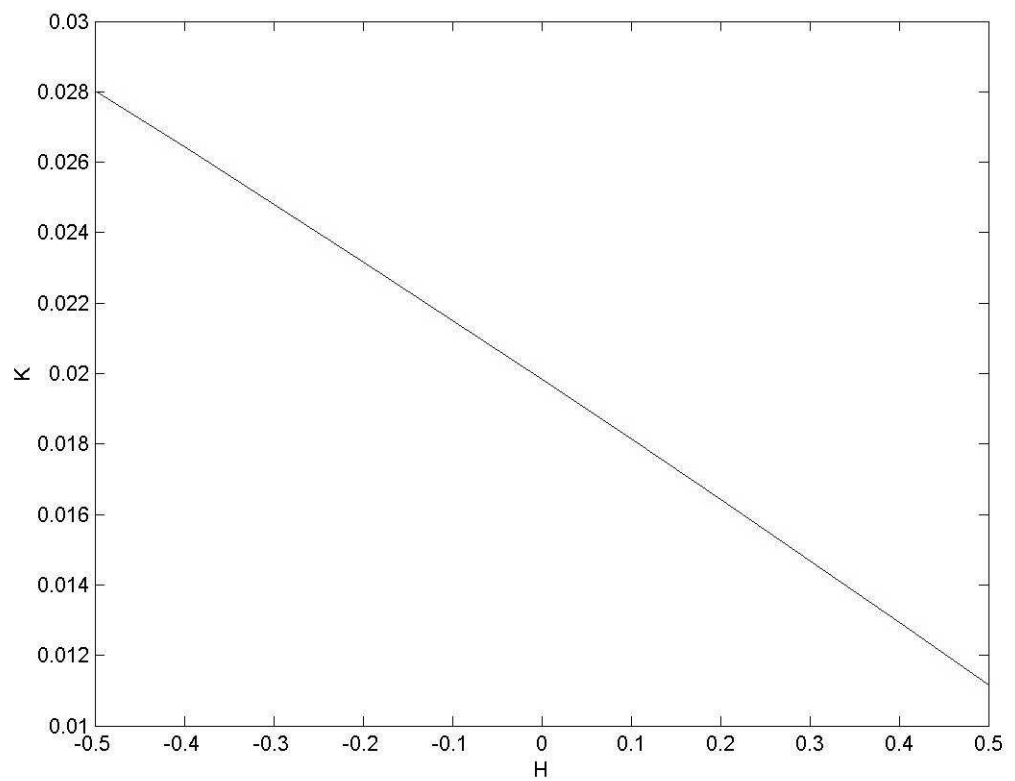
These graphs show that the strategy of investing more when the interest rates are low increases the expected return of the portfolio. Expected covariance of portfolio value with factors increases for large H . Surprisingly, the volatility of the portfolio is minimized at a certain non-zero level of H .

For the second illustration, assume that the target function (6) is specified with factor-sensitivity $\Gamma = 0$. Then the dependence of optimal (h, H) combination on the risk-sensitivity parameter θ is illustrated on Figure 4.

This picture shows that the increase in the risk-sensitivity leads to smaller average investment in stocks and to smaller sensitivity of the investment to changes in the interest rates. It is interesting to note, however, that the ratio of H to h , which can be interpreted as the relative sensitivity of investments to interest rates is almost constant for sufficiently large values of risk-sensitivity.

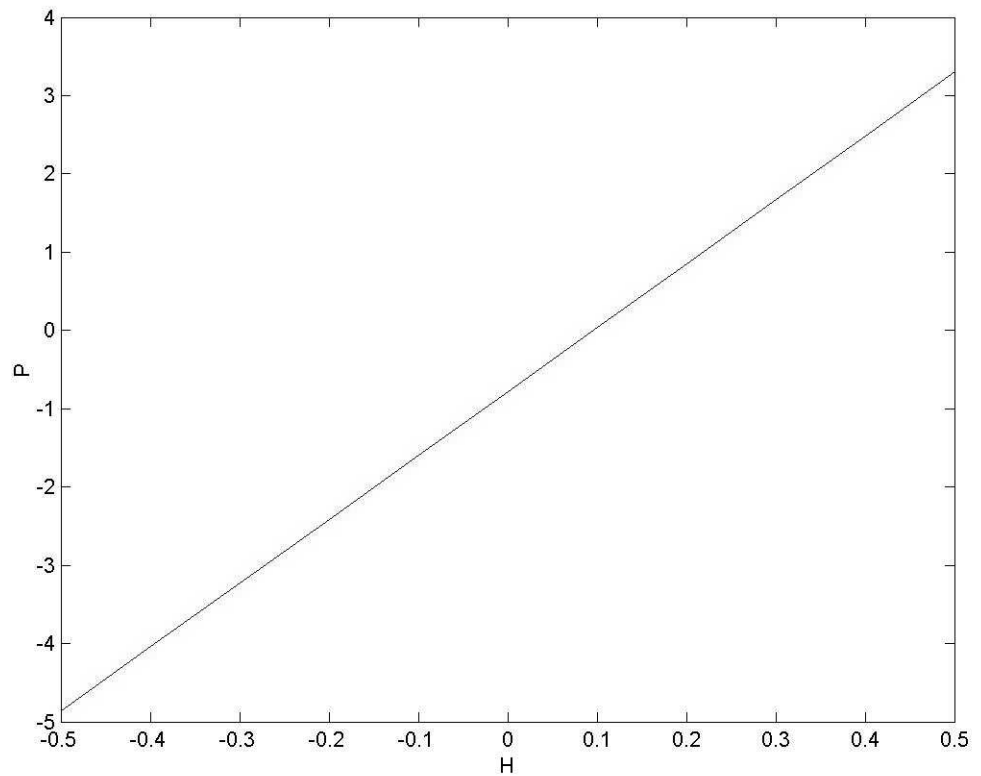
Now consider factor-sensitivity parameter Γ changing from 0 to 0.01. The graph on Figure 5 shows the optimal (h, H) path. The greater sensitivity of the investment rule to the covariance of the portfolio value with the factor leads to smaller average investment and smaller sensitivity of the investment rule to the factor. It is interesting, however, that the sensitivity of investment rule to factor is affected smaller than the average investment amount.

Figure 1: Dependence of $K = E(du)$ from H
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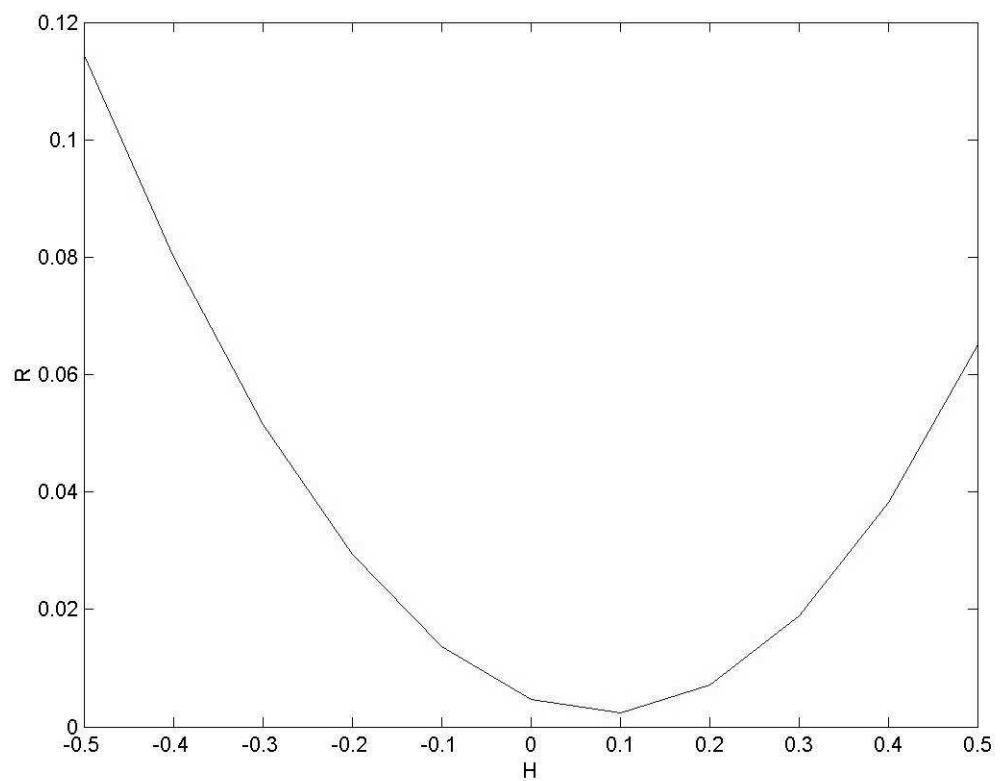
K is the expectation of portfolio monthly return; H is the sensitivity of an investment rule to the factor; the factor is 3-month Treasury Bill rate measured in percentage terms.

Figure 2: Dependence of $P = \lim_{t \rightarrow \infty} E(Xu)$ from H
Figure 2. Dependence of $P = \lim_{t \rightarrow \infty} E(Xu)$ from H .



P is the long-run covariance of the portfolio monthly return and the factor; H is the sensitivity of an investment rule to the factor; the factor is 3-month Treasury Bill rate measured in percentage terms.

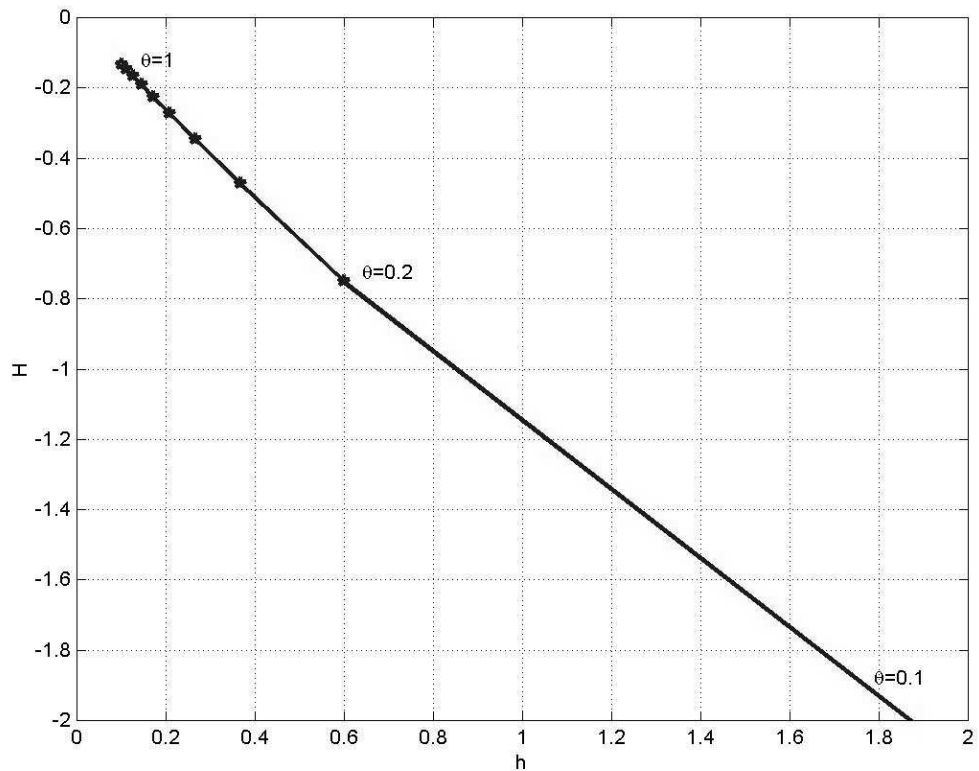
Figure 3: Dependence of $R = \lim_{t \rightarrow \infty} (1/t) \text{Var}(u)$ from H
Figure 3. Dependence of $R = \lim_{t \rightarrow \infty} (1/t) \text{Var}(u)$ from H .



R is the long-run growth rate in variance of the portfolio value; H is the sensitivity of an investment rule to the factor; the factor is 3-month Treasury Bill rate measured in percentage terms.

Figure 4: Dependence of optimal (h, H) -strategy from risk-sensitivity parameter

θ
Figure 4. Dependence of optimal (h, H) -strategy from risk-sensitivity parameter θ .

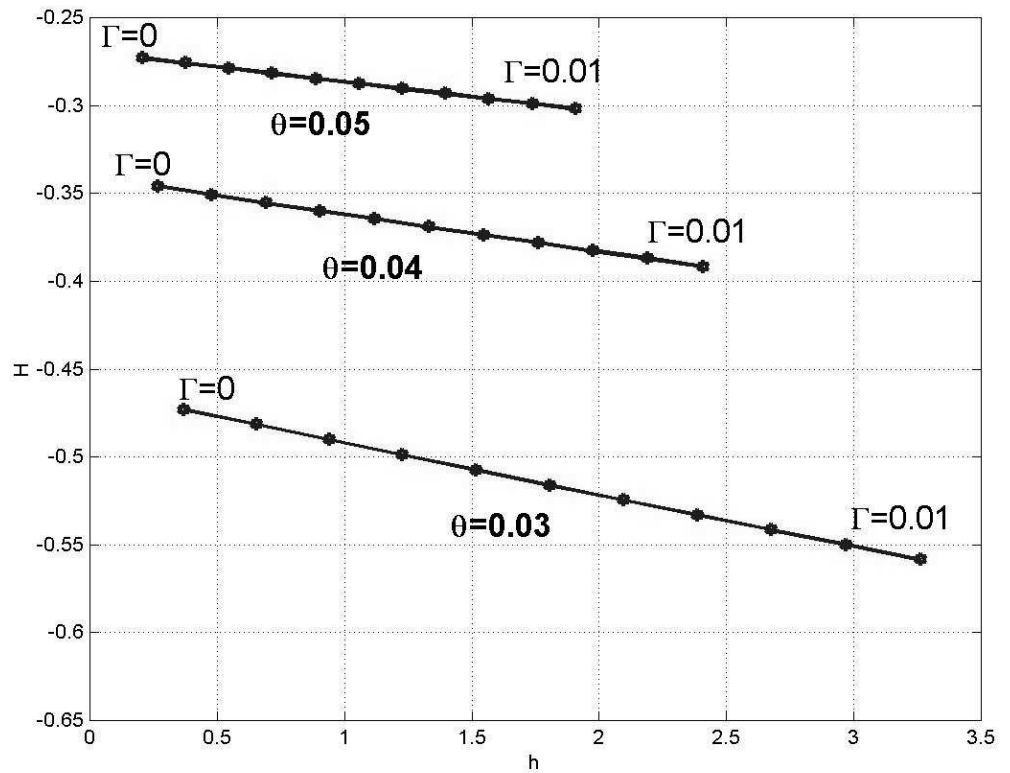


$\Gamma=0$; H is the sensitivity of an investment rule to the factor; h is the investment amount if the deviation of the factor from its average value is 0; ; the factor is 3-month Treasury Bill rate measured in percentage terms.

Figure 5: Dependence of optimal (h, H) -strategy from risk-sensitivity parameter

θ and factor-sensitivity parameter Γ

Figure 5: Dependence of optimal (h, H) -strategy from risk-sensitivity parameter θ and factor-sensitivity parameter Γ .



5 Conclusion

This paper derives explicit formulas for asymptotic joint moments of the logarithm of investor's wealth and factors, allowing for a fast maximization of linear combinations of the moments. An application of this method to the real data shows that the optimal strategy is very sensitive to the factor-sensitivity parameter Γ that measures dependence of the investment criterion on the covariance of wealth with factors.

A Proof of Theorem 1

First, we are going to compute Eu :

$$\begin{aligned} du &= d(\ln U) = \frac{dU}{U} - \frac{1}{2}\left(\frac{dU}{U}\right)^2 \\ &= (h + HX)'((a + AX)dt + \Sigma dW) \\ &\quad - \frac{1}{2}(h + HX)'\Sigma\Sigma'(h + HX)dt; \end{aligned} \tag{17}$$

$$Edu = (h'a + \text{tr}(A\Delta H'))dt - \frac{1}{2}(h'\Sigma\Sigma'h + \text{tr}(\Sigma'H\Delta H'\Sigma))dt. \tag{18}$$

Integrating (18) over time gives

$$Eu = Kt, \tag{19}$$

where

$$K =: h'a + \text{tr}(A\Delta H') - \frac{1}{2}(h'\Sigma\Sigma'h + \text{tr}(\Sigma'H\Delta H'\Sigma)). \tag{20}$$

Our second goal is to compute $E(uX)$.

$$\begin{aligned}
d(uX) &= Xdu + udX + dudX \\
&= X[(h + HX)'((a + AX)dt + \Sigma dW) - \frac{1}{2}(h + HX)'\Sigma\Sigma'(h + HX)dt] \\
&\quad + u(BXdt + \Lambda dW) \\
&\quad + [(h + HX)'((a + AX)dt + \Sigma dW) - \frac{1}{2}(h + HX)'\Sigma\Sigma'(h + HX)dt](BXdt + \Lambda dW)
\end{aligned} \tag{21}$$

Using $E(X) = E(X^3) = 0$, we get

$$Ed(uX) = dt(\Delta A'h + \Delta H'a - \Delta H'\Sigma\Sigma'h + BE(uX) + \Lambda\Sigma'h) \tag{22}$$

Denoting $E(uX)$ as $P(t)$, we note that it satisfies the following differential equation

$$\frac{dP}{dt} = BP + \Delta A'h + \Delta H'a - \Delta H'\Sigma\Sigma'h + \Lambda\Sigma'h. \tag{23}$$

Since B is stable,

$$P(t) \rightarrow B^{-1}[\Delta(H'\Sigma\Sigma'h - A'h - H'a) - \Lambda\Sigma'h] \text{ as } t \rightarrow \infty. \tag{24}$$

Thus, we computed $P =: \lim_{t \rightarrow \infty} P(t)$, which is the asymptotic covariance of the portfolio and factors.

The next step is to compute $E(uXX')$.

$$\begin{aligned}
d(uXX') &= duXX' + u(dX)X' + uXdX' + dudXX' + (du)XdX' + udXdX' \\
&= [(h + HX)'((a + AX)dt + \Sigma dW) - \frac{1}{2}(h + HX)'\Sigma\Sigma'(h + HX)dt]XX' \\
&\quad + u(BXdt + \Lambda dW)X' \\
&\quad + uX(BXdt + \Lambda dW)' \\
&\quad + [(h + HX)'((a + AX)dt + \Sigma dW) - \frac{1}{2}(h + HX)'\Sigma\Sigma'(h + HX)dt](BXdt + \Lambda dW)X' \\
&\quad + [(h + HX)'((a + AX)dt + \Sigma dW) - \frac{1}{2}(h + HX)'\Sigma\Sigma'(h + HX)dt]X(BXdt + \Lambda dW)' \\
&\quad + u(BXdt + \Lambda dW)(BXdt + \Lambda dW)'
\end{aligned} \tag{25}$$

$$\begin{aligned}
Ed(uXX') &= dt[h'a\Delta + E(\text{tr}(AXX'H')XX') \\
&\quad - \frac{1}{2}h'\Sigma\Sigma'h\Delta - \frac{1}{2}E(\text{tr}(\Sigma' HXX'H'\Sigma)XX')] \\
&\quad + BE(uXX') + E(uXX')B' + 2\Lambda\Sigma'H\Delta + E(u)\Lambda\Lambda'
\end{aligned} \tag{26}$$

Asymptotically, $E(uXX') \sim Rt + S$, and the coefficients R, S can be found from the equation:

$$\begin{aligned}
R &= h'a\Delta + E(\text{tr}(AXX'H')XX') - \frac{1}{2}h'\Sigma\Sigma'h\Delta - \frac{1}{2}E(\text{tr}(\Sigma' HXX'H'\Sigma)XX') \\
&\quad + B(Rt + S) + (Rt + S)B' + 2\Lambda\Sigma'H\Delta + K\Lambda\Lambda't,
\end{aligned} \tag{27}$$

which reduces to two equations on R and S ,

$$0 = BR + RB' + K\Lambda\Lambda', \quad (28)$$

$$\begin{aligned} R &= h'a\Delta + E(\text{tr}(AXX'H')XX') - \frac{1}{2}h'\Sigma\Sigma'h\Delta - \frac{1}{2}E(\text{tr}(\Sigma'HXX'H'\Sigma)XX') \\ &\quad + BS + SB' + 2\Lambda\Sigma'H\Delta. \end{aligned} \quad (29)$$

Since Δ satisfies $B\Delta + \Delta B' + \Lambda\Lambda' = 0$, (28) implies

$$R = K\Delta \quad (30)$$

and, therefore, (29) can be rewritten as

$$\begin{aligned} K\Delta &= h'a\Delta + E(\text{tr}(AXX'H')XX') - \frac{1}{2}h'\Sigma\Sigma'h\Delta - \frac{1}{2}E(\text{tr}(\Sigma'HXX'H'\Sigma)XX') \\ &\quad + BS + SB' + 2\Lambda\Sigma'H\Delta. \end{aligned} \quad (31)$$

Using the identity

$$E(x_i x_j x_k x_l) = \Delta_{ij}\Delta_{kl} + \Delta_{ik}\Delta_{jl} + \Delta_{il}\Delta_{jk}, \quad (32)$$

we can further reduce (31) to

$$BS + SB' = -2\Delta H'A\Delta + \Delta H'\Sigma\Sigma'H\Delta - 2\Lambda\Sigma'H\Delta. \quad (33)$$

For later use, we also need to compute $E(uX'QX)$ where Q is an arbitrary matrix. Because of the identity

$$E(uX'QX) = \text{tr}(E(uXX')Q), \quad (34)$$

we have

$$E(uX'QX) \sim \text{tr}(RQ)t + \text{tr}(SQ) \text{ when } t \rightarrow \infty \quad (35)$$

The next step is to compute $E(u^2)$.

$$\begin{aligned}
d(u^2) &= 2u(du) + (du)^2 \\
&= 2u[(h + HX)'((a + AX)dt + \Sigma dW) - \frac{1}{2}(h + HX)'\Sigma\Sigma'(h + HX)dt] \\
&\quad + (h + HX)'\Sigma\Sigma'(h + HX)dt
\end{aligned} \tag{36}$$

$$\begin{aligned}
Ed(u^2) &= 2dt[h'aEu + E(uX')H'a + h'AE(uX) + E(uX'H'AX)] \\
&\quad - \frac{1}{2}(h'\Sigma\Sigma'hE(u) + h'\Sigma\Sigma'HE(uX) + E(uX')H'\Sigma\Sigma'h + E(uX'H'\Sigma\Sigma'HX)) \\
&\quad + E((h + HX)'\Sigma\Sigma'(h + HX))] \\
&= 2dt[h'aKt + P'H'a + h'AP + \text{tr}(RH'A)t + \text{tr}(SH'A)] \\
&\quad - \frac{1}{2}(h'\Sigma\Sigma'hKt + h'\Sigma\Sigma'HP + P'H'\Sigma\Sigma'h + \text{tr}(RH'\Sigma\Sigma'H)t + \text{tr}(SH'\Sigma\Sigma'H)) \\
&\quad + \frac{1}{2}E((h + HX)'\Sigma\Sigma'(h + HX))]
\end{aligned} \tag{37}$$

From this we have,

$$\begin{aligned}
Eu^2 &= \text{const} + t^2[h'aK + \text{tr}(RH'A) - \frac{1}{2}(h'\Sigma\Sigma'hK + \text{tr}(RH'\Sigma\Sigma'H))] \\
&\quad + t[2(P'H'a + h'AP + \text{tr}(SH'A)) - (h'\Sigma\Sigma'HP + P'H'\Sigma\Sigma'h + \text{tr}(SH'\Sigma\Sigma'H))] \\
&\quad + E((h + HX)'\Sigma\Sigma'(h + HX)].
\end{aligned} \tag{38}$$

Finally,

$$\begin{aligned}
Var(u) &= E(u^2) - E(u)^2 \\
&= \text{const} + 2(P'H'a + h'AP + \text{tr}(SH'A))t \\
&\quad - (h'\Sigma\Sigma'HP + P'H'\Sigma\Sigma'h + \text{tr}(SH'\Sigma\Sigma'H))t \\
&\quad + (h'\Sigma\Sigma'h + \text{tr}(\Sigma'H\Delta H'\Sigma))t
\end{aligned} \tag{39}$$

Using the definition of P this can be manipulated into

$$\begin{aligned}
Var(u) &= t[((h'\Sigma\Sigma'H - h'A - a'H)B^{-1}\Lambda + h'\Sigma)((h'\Sigma\Sigma'H - h'A - a'H)B^{-1}\Lambda + h'\Sigma)' \\
&\quad + \text{tr}(2SH'A + (\Delta - S)H'\Sigma\Sigma'H)] + \text{const}
\end{aligned} \tag{40}$$

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